New Implementation of High-Gain Observers in the Presence of Measurement Noise Using Stochastic Approximation

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Abstract— **In this paper, we propose a new method for implementation of a High-Gain Observer (HGO) to cope with measurement noise through Stochastic Approximation (SA). Most approaches have been using a trade-off between a speed of regeneration of states in the presence of uncertainties and the effect of measurement noise. Through SA, we are able to reserve authentic properties of the HGO i.e., a fast reconstruction of system states and robustness to uncertainties, while dealing with the effect of measurement noise both in practical and analytical manners. Regularity conditions are verified to analyze the stability by the associated Ordinary Differential Equation (ODE), i.e., the ODE method. Stability analysis of the full system is conducted. Numerical simulation results demonstrate the validity of the proposed design method.**

I. INTRODUCTION

It is well-known in [1], [2], [3] that HGOs have properties, i.e., quick convergence and performance recovery with high gains while high gains amplify the effect of measurement noise. There is a trade-off between fast reconstruction of system states under uncertainties and the effect of measurement noise. Based on this trade-off, high gains are used in a transient period to obtain a fast regeneration of states since the effect of measurement noise is small enough in an innovation term. In the steady state period, relatively low gains are used to reduce the effect of measurement noise because the effect of measurement noise is not negligible in the innovation term. To deal with this trade-off, Ahrens and Khalil in [1] implemented switched high gains where high gain is used during the transient period and relatively low gain is used at steady state. In [2], an adaptive law was applied to appropriately set high-gain values in the HGO to have the trade-off. Prasov and Khalil in [3] employed nonlinear high gains in the HGO to achieve the trade off. All the aforementioned schemes were realized in continuous-time systems. In [4], without using high gains the modulation integral observer for linear continuous- and discrete-time Single-Input-Single-Output (SISO) systems in the presence of bounded additive measurement noise was introduced based on modulation integrals. The modulation integrals can deal with hybrid dynamic systems, i.e, linear continuous- and discrete-time systems.

The ODE method (see [5], [6], and [7]) for SA algorithms has been established with regularity conditions in order to obtain averaged dynamics without the effect of noise. Applications of SA were introduced in design of controls for multi-agent systems under noise in [8], [9], [10]. In [8], through the SA method, the effect of measurement noise was averaged out and stability of the closed-loop system was guaranteed. The effect of noise in [9] was coped with in optimization for a mobile robotic network. Authors in [10] employed the SA approach for stochastic source seeking with mobile robot networks and showed the experimental validation.

This paper introduces a new way to implement the HGO in the presence of measurement noise via SA. Instead of using the trade-off between a fast reconstruction of states under uncertainties and the effect of measurement noise, SA is used to reserve a fast reconstruction of states with robustness to uncertainties and to deal with measurement noise in both practical and analytical senses. Practically, we show that the effect of measurement noise was diminished. Most importantly, our scheme enables us to analyze the stability of the system in the absence of the measurement noise via the ODE method. The regularity conditions are checked to use the ODE method [5]. Stability analysis of the full system is fulfilled. Through numerical simulation, the effectiveness of the proposed method is demonstrated.

This paper is organized as follows. In Section II, the SA scheme is reviewed. In Section III, the problem statement is given. Section IV presents a new method for implementation of the HGO to average out the effect of measurement noise using the ODE method. In Section V, the stability of the full system is analyzed through regularity conditions in the ODE method. The simulation results are shown in Section VI to demonstrate the validity of the proposed method. Section VII gives concluding remarks and future work.

II. Stochastic Approximation

In this section, we introduce the Stochastic Approximation [5] in order to use the ODE method [5], [6], and [7], i.e., the stochastic difference equation asymptotically tracks a deterministic ODE.

The SA scheme considers the discrete-time system

$$x(k+1) = x(k) + a(k)[h(x(k)) + M(k)], \quad k \geq 0 \tag{1}$$

where $x, \ h \in \mathbf{R}^r$; k is the kth sampling instant time, $t = \sum_{k=1}^{n} a(k)$ with a decreasing sequence (decreasing sampling period) $a(k)$, and $M(k)$ is a Martingale Difference Sequence (MDS). To use the ODE method, the system (1) is required to satisfy the following conditions.

(A.1) The map $h : \mathbf{R}^r \to \mathbf{R}^r$ is Lipschitz: $\|h(a) - h(b)\| \leq L\|a - b\|$ for some $0 < L < \infty$.

(A.2) Stepsizes $a(k)$ are positive scalars satisfying

$$\sum_{k=1} a(k) = \infty, \quad \sum_{k=1} a^2(k) < \infty \tag{2}$$

(A.3) $M(k)$ is a martingale difference sequence with respect to the increasing family of σ-fields

$$\mathscr{F}_k \triangleq \sigma(x(m), M(m), m \leq k) \tag{3}$$

for $k \geq 0$, such that

$$E[M(k+1)|\mathscr{F}_k] = 0, \quad \text{Almost Surely (a.s.)}, \tag{4}$$

for $k \geq 0$, and $M(k)$ are square-integrable with

$$E[\|M(k+1)\|^2|\mathscr{F}_k] \leq K(1 + \|x(k)\|^2) \ \text{a.s.} \tag{5}$$

for $k \geq 0$ and some constant $K > 0$.

(A.4) The iterates of (1) remain bounded a.s., that is

$$\sup_k \|x(k)\| < \infty, \quad \text{a.s.} \tag{6}$$

Through the next Lemma 2.1 and Theorem 2.2 (i.e., respective Lemma 1 and Theorem 2 in Chapter 2.1 of [5], i.e., the ODE method), the stochastic difference equation (1) asymptotically tracks a deterministic ODE

$$\dot{x}(t) = h(x(t)), \quad t \geq 0 \tag{7}$$

The main idea for the Lemma is to construct an interpolated trajectory $\bar{x}(t)$ during a time period $I_k = [t(k), t(k+1)]$, $k \geq 0$. Then, it is shown that the interpolated variable $\bar{x}(t)$ almost surely approaches the solution set of the ODE (7). Let us define a piecewise, continuous linear $\bar{x}(t)$, $t \geq 0$, by $\bar{x}(t(k)) = x(k)$, $k \geq 0$, with linear interpolation on each interval I_k

$$\bar{x}(t) = x(k) + [x(k+1) - x(k)]\frac{t - t(k)}{t(k+1) - t(k)}, \quad t \in I_k \tag{8}$$

Let $x^s(t)$ $t \geq s$, denote the unique solution of (7) starting at the instant time s

$$\dot{x}^s = h(x^s(t)), \quad t \geq s, \tag{9}$$

with $x^s = \bar{x}(s)$ $s \in \mathbf{R}$. Similarly, let x_s, $t \leq s$, denote the unique solution of (7) ending at the instant time s

$$\dot{x}_s = h(x_s(t)), \quad t \leq s, \tag{10}$$

with $x_s(s) = \bar{x}(s), s \in \mathbf{R}$.

Lemma 2.1: For any $T > 0$,

$$\begin{aligned} \lim_{s\to\infty} \sup_{t\in[s,s+T]} \|\bar{x}(t) - x^s(t)\| = 0, \quad \text{a.s.} \\ \lim_{s\to\infty} \sup_{t\in[s-T,s]} \|\bar{x}(t) - x_s(t)\| = 0, \quad \text{a.s.} \end{aligned} \tag{11}$$

Theorem 2.2: Almost surely, the sequence $\{x(k)\}$ generated by (1) converges to a (possibly sample path dependent) compact connected internally chain transitive invariant set of (7).

III. Problem Statement

We consider a continuous differential system, i.e., a single-input-single-output nonlinear system

$$\dot{x} = A_c x + B_c \phi(x) \triangleq f(x) \tag{12}$$

$$y = C_c x \tag{13}$$

where $x \in \mathbf{R}^r$ is the state, $y \in \mathbf{R}$ is the measured output, and the matrices $A_c \in \mathbf{R}^{r\times r}$, $B_c \in \mathbf{R}^{r\times 1}$, and $C_c \in \mathbf{R}^{1\times r}$ are

$$A_c = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \ B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{14}$$
$$C_c = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \end{bmatrix}$$

Assumption 1:

- The function $\phi(x)$ is globally Lipschitz and $\phi(0) = 0$.
- The system (12) is asymptotically stable at $x = 0$.

For the system (12), the High-Gain Observer is given by

$$\begin{aligned} \dot{\hat{x}} &= A_c\hat{x} + Hy - HC_c\hat{x} \\ &= A_c\hat{x} + HC_c(x - \hat{x}) \end{aligned} \tag{15}$$

where $\hat{x} \in \mathbf{R}^r$ is the estimate of x and the observer gain H is

$$H = \begin{bmatrix} \frac{h_1}{\varepsilon} & \frac{h_2}{\varepsilon^2} & \dots & \dots & \frac{h_r}{\varepsilon^r} \end{bmatrix}^T \tag{16}$$

Sufficiently small $\varepsilon > 0$ is used and h_i for $i = 1, \dots, r$ are chosen such that the polynomials

$$s^r + h_1 s^{r-1} + \cdots + h_r = 0 \tag{17}$$

are Hurwitz.

It was shown in [1] that the high gain H in (16) plays an important role to attenuate uncertainties whereas it amplifies the effect of measurement noise.

IV. Implementation of High-Gain Observers

We propose to use SA [5] to deal with the effect of measurement noise while utilizing the high gains to deal with uncertainties.

For design purpose, we consider the following continuous-time HGO

$$\begin{aligned} \dot{q} &= \left(\frac{1}{\varepsilon}\right)[A_c q + H_0(y - C_c q)] \\ &= \left(\frac{1}{\varepsilon}\right)[A_0 q + H_0 y] \end{aligned} \tag{18}$$

where ε is a small positive parameter,

$$\begin{aligned} & q = D\hat{x} \\ & D = \text{diag}[1\ \varepsilon\ \cdots\ \varepsilon^{r-1}] \\ & A_0 = A_c - H_0 C_c \\ & H_0 = \begin{bmatrix} h_1 & h_2 & \ldots & \ldots & h_r \end{bmatrix}^T \end{aligned} \tag{19}$$

and h_i for $i = 1, \ldots, r$ are chosen such that A_0 is Hurwitz. The earlier works on HGO implementation with a fixed sampling time period were presented in [11], [12], [13]. The HGO in (18) is discretized using the Forward Difference (FD) to obtain

$$q(k+1) = q(k) + \left(\frac{T}{\varepsilon}\right)\Big[A_0 q(k) + H_0 y(k)\Big] \tag{20}$$

where the measurement y is $y = x_1 + \nu$, T is a fixed sampling time period, and $\nu(k)$ is measurement noise and has the following assumption.

Assumption 2: The measurement noise $\nu(k)$ is bounded independent identically distributed random variable with zero mean and finite variance.

We note that $\nu(k)$ is a MDS, since $\nu(k)$ is a bounded independent random variable with zero mean and finite variance in Chapter 3.3 of [14].

The key idea of our SA scheme is to use the decreasing sequence $a(k)$

$$a(k) = \alpha\varepsilon b(k) \tag{21}$$

where $b(k) < 1$ is chosen to satisfy (2) in (A.2) and α is chosen to be sufficiently small enough, which will be described in the next section. With the decreasing sequence $a(k)$, the HGO is implemented as

$$\begin{aligned} q(k+1) &= q(k) + \frac{a(k)}{\varepsilon}[A_0 q(k) + H_0 y(k)] \\ &= q(k) + \frac{a(k)}{\varepsilon}[A_0 q(k) + H_0 C_c x(k) + H_0 \nu(k)] \\ \hat{x}(k) &= D^{-1} q(k) \end{aligned} \tag{22}$$

V. Convergence Analysis

In this section, we check regularity conditions (A.1) to (A.4) in Section II for implementing the HGO in (22) to use Lemma 2.1 and Theorem 2.2.

To obtain a discrete-time model of the full system, the plant is discretized using the same scheme that was used to discretize the observer. Using the FD, the plant dynamics (12) is discretized as

$$x(k+1) = x(k) + a(k)[A_c x(k) + B_c \phi(x(k))] \tag{23}$$

To obtain the MDS satisfying (A.3), we follow the procedure in Chapter 3.3, [5]. With (22) and (23), the full system is obtained by

$$\chi(k+1) = \chi(k) + a(k)\mathbf{F}(\chi(k), \nu(k)) \tag{24}$$

where

$$\chi = [x\ \ q]^T, \quad \mathbf{F}(\chi, \nu) = \begin{bmatrix} A_c x + B_c \phi(x) \\ \frac{1}{\varepsilon}[A_0 q + H_0 C_c x + H_0 \nu] \end{bmatrix} \tag{25}$$

Let us take the expectation

$$h(\chi) = E[\mathbf{F}(\chi, \nu)] \tag{26}$$

For the condition (A.1), $h(\chi)$ satisfies Lipschitz condition since $\phi(x)$ is Lipschitz with respect to x from Assumption 1. The decreasing sequence $a(k)$ for the SA scheme is chosen to satisfy the condition (A.2). A MDS $\bar{\nu}(k)$ is defined by

$$\bar{\nu}(k) = \mathbf{F}(\chi(k), \nu(k)) - h(\chi(k)) = \begin{bmatrix} 0 \\ \frac{H_0}{\varepsilon}\nu(k) \end{bmatrix} \tag{27}$$

which satisfies (4) in (A.3).

For the SA, we rewrite the full system as

$$\chi(k+1) = \chi(k) + a(k)[h(\chi(k)) + \bar{\nu}(k)] \tag{28}$$

To check the condition (A.4), we note that the states of the plant are bounded by Assumption 1. We show boundedness of the state of the observer. An equation of the second row in (28) is written as

$$q(k+1) = q(k) + \alpha b(k)\,[A_0 q(k) + H_0 G(k)] \tag{29}$$

where $G(k)$

$$G(k) = C_c x(k) + \nu(k) \tag{30}$$

is viewed as a bounded input of the system (29) due to boundedness of the state $x(k)$ and MDS $\bar{\nu}(k)$. For the condition (A.4), we need to show that the state of the system (29) is bounded. Let $V = q^T P q$ and P is a solution of the Lyapunov equation

$$A_0^T P + P A_0 = -Q \tag{31}$$

with $Q \succ 0$, where symbols $\succ$ and $\prec$ denote positive and negative definite matrices, respectively. Let us take $\Delta V = V(k+1) - V(k)$.

$$\begin{aligned} \Delta V &= q^T(k+1)Pq(k+1) - q^T(k)Pq(k) \\ &= \{q(k) + \alpha b(k)[A_0 q(k) + BG(k)]\}^T p \\ &\quad \times \{q(k) + \alpha b(k)[A_0 q(k) + BG(k)]\} - q^T(k)Pq(k) \end{aligned} \tag{32}$$

Then

$$\Delta V \leq -\alpha b(k)\Big[c_1 - \alpha b(k) M_p\Big]\|q(k)\|^2 + \alpha b(k)\big[c_2 + \alpha b(k) c_3\big]\|q(k)\| + \alpha^2 b^2(k) c_4 \tag{33}$$

where

$$M_p = \|A_0\|^2 \|P\| \tag{34}$$

$\|A_0\| = \sqrt{\lambda_{\max}(A_0^T A_0)}$, $\|P\| = \lambda_{\max}(P)$, $\lambda_{\max}(N)$ denotes a maximum eigenvalue of N, and c_1, c_2, c_3, and c_4 are positive constants independent of α and $b(k)$. ΔV can be rewritten as

$$\Delta V \leq -\alpha b(k)\Big[c_5 - \alpha b(k) c_6\Big]V(k) + \alpha b(k)\big[c_7 + \alpha b(k) c_8\big]\sqrt{V}(k) + \alpha^2 b^2(k) c_4 \tag{35}$$

with c_5, c_6, c_7, $c_8 > 0$, which shows that with sufficiently small enough α, the system is Bounded-Input-Bounded-State (BIBS) stable.

Remark 1: To find a less conservative value of α, a feasibility Linear Matrix Inequality (LMI) can be used for BIBS stability, which is equivalent to the condition

$$\bar{A}^T P \bar{A} - P = \alpha b(k)[A_0^T P + P A_0 + \alpha b(k) A_0^T P A_0] \prec -cI \tag{36}$$

where $\bar{A} = I + \alpha b(k) A_0$ and $c > 0$. For the selected α and $b(k)$, there should be $P = P^T \succ 0$ such that (36) is satisfied.

All regularity conditions (A.1) to (A.4) are satisfied. Using Lemma 2.1 and Theorem 2.2 in Section II, now we can analyze the stochastic difference equation (24) through the associated ODE

$$\dot{\chi} = h(\chi) \tag{37}$$

where measurement noise is averaged out. The stability of the averaged dynamics (37) of the full system can be analyzed similar to [15]. From (37), we conclude that the averaged dynamics of the plant is asymptotically stable at $x = 0$ and the averaged dynamics of the HGO is exponentially stable at $q = 0$.

VI. Simulation Results

In this section, numerical simulation results in HGOs with the fixed sampling time period T and the decreasing sequence $a(k)$ are compared.

The plant is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{x_1}{1 + x_1^2} - x_2 \end{aligned} \tag{38}$$

The continuous-time HGO for the plant is

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \frac{h_1}{\varepsilon}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \frac{h_2}{\varepsilon^2}(y - \hat{x}_1) \end{aligned} \tag{39}$$

where y is a measurement, $\hat{x}_1$ and $\hat{x}_2$ are estimates of x_1 and x_2, respectively, and the observer gains h_1 and h_2 are chosen such that the polynomials

$$s^2 + h_1 s + h_2 = 0 \tag{40}$$

is Hurwitz. Using the FD and the fixed sampling time period T, the plant and the HGO are discretized as

$$\begin{aligned} x_1(k+1) &= x_1(k) + T x_2(k) \\ x_2(k+1) &= x_2(k) + T\left[-\frac{x_1(k)}{1 + x_1^2(k)} - x_2(k)\right] \\ y(k) &= x_1(k) + \nu(k) \end{aligned} \tag{41}$$

where the fixed sampling time T is chosen as $T = 0.0004$, and the measurement noise $\nu(k)$ is the i.i.d. uniform on $[-0.5\ 0.5]$, and

$$\begin{aligned} q_{1_f}(k+1) &= q_{1_f}(k) + \frac{T}{\varepsilon}[q_{2_f}(k) + h_1(y(k) - q_{1_f}(k))] \\ q_{2_f}(k+1) &= q_{2_f}(k) + \frac{T}{\varepsilon}[h_2(y(k) - q_{1_f}(k))] \\ \hat{x}_1(k) &= q_{2_f}(k), \quad \hat{x}_2(k) = \frac{q_{2_f}(k)}{\varepsilon} \end{aligned} \tag{42}$$

where

$$h_1 = 2, \quad h_2 = 2, \quad \varepsilon = 0.01, \quad \alpha = 0.04 \tag{43}$$

The HGOs with SA for (41) are

$$\begin{aligned} q_1(k+1) &= q_1(k) + \frac{a(k)}{\varepsilon}[q_2(k) + h_1(y(k) - q_1(k))] \\ q_2(k+1) &= q_2(k) + \frac{a(k)}{\varepsilon}[h_2(y(k) - q_1(k))] \\ \hat{x}_1(k) &= q_1(k), \quad \hat{x}_2(k) = \frac{q_2(k)}{\varepsilon} \end{aligned} \tag{44}$$

where $a(k) = \alpha\varepsilon b(k)$ and $b(k) = 1/(1 + k/20000)$ is chosen to satisfy the conditions in (2) with the same values of h_1, h_2, α, and ε in the fixed sampling HGO. Initial conditions for (41) are

$$x_1(0) = 10, \quad x_2(0) = 10 \tag{45}$$

Initial conditions for the fixed sampling HGOs are

$$q_{1_f}(0) = 0, \quad q_{2_f}(0) = 0 \tag{46}$$

Initial conditions for the HGOs with SA are

$$q_1(0) = 0, \quad q_2(0) = 0 \tag{47}$$

In Figs. 1 and 2, the plant states x_1 and x_2 generated by the deterministic ODE (38), and estimates $\hat{x}_1$ and $\hat{x}_2$ generated by the fixed sampling HGO, and $\hat{x}_1$ and $\hat{x}_2$ generated by the HGO with SA are plotted.

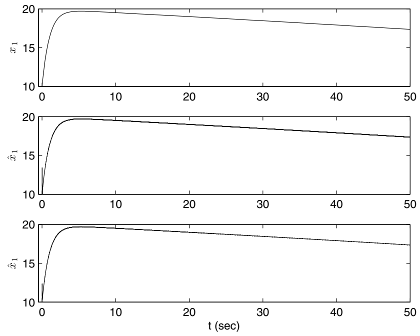

Fig. 1. Trajectories x_1 (the top) generated by the deterministic ODE (38), $\hat{x}_1$ (the middle) generated by the fixed sampling HGO, and $\hat{x}_1$ (the bottom) generated by the HGO with SA.

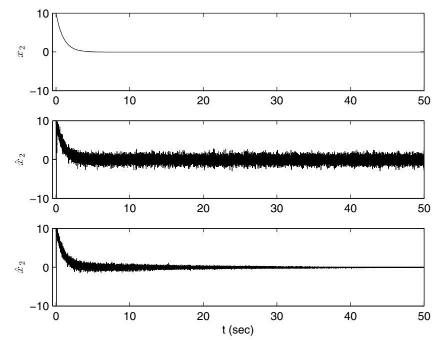

Fig. 2. Trajectories x_2 (the top) generated by the deterministic ODE (38), $\hat{x}_2$ (the middle) generated by the fixed sampling HGO, and $\hat{x}_2$ (the bottom) generated by the HGO with SA.

Figs. 1 and 2 show that trajectories of the solution of the stochastic difference equation with measurement noise asymptotically follows trajectories of the solution of the deterministic ODE. Figs. 3 and 4 show the estimation error $e_1 = x_1 - \hat{x}_1$ generated by the fixed-sampling and SA, respectively. Figs. 5 and 6 show the estimation error $e_2 = x_2 - \hat{x}_2$ generated by the fixed-sampling and SA, respectively. The error trajectories in Figs. 4 and 6 are decreasing while the error trajectories in Figs. 3 and 5 remain with non-decreasing fluctuations.

VII. Conclusions and Future Work

The HGOs have intrinsic merits: a fast regeneration of states and robustness to uncertainties through high gains. However, under measurement noise, the high gains degrade the performance of the HGOs due to amplifying the measurement noise. In this paper, while reserving the intrinsic merits of the HGOs, the effect of measurement noise was coped with by using SA. Regularity conditions for the ODE method were shown to be satisfied. Stability analysis of the full system was fulfilled. Simulation results support the validity of the proposed method.

For future work, we will investigate the HGO under output feedback control and analyze stability of the closed-loop system.

Acknowledgment

This work was supported in part by the National Science Foundation under grant numbers ECCS-1128476 and CAREER award CMMI-0846547.

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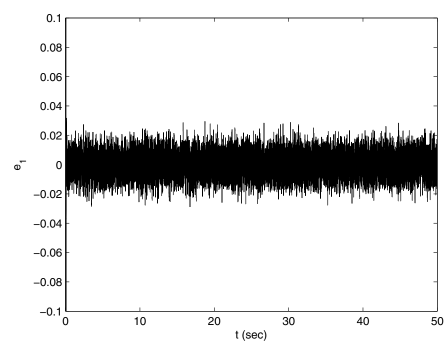

Fig. 3. An error $e_1 = x_1 - \hat{x}_1$ with fixed sampling time

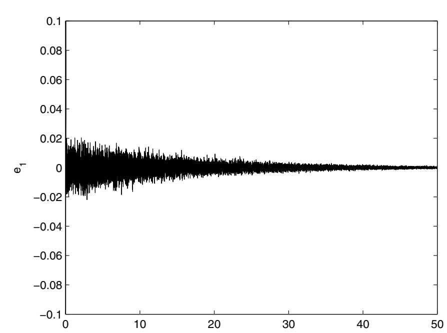

Fig. 4. An error $e_1 = x_1 - \hat{x}_1$ with SA

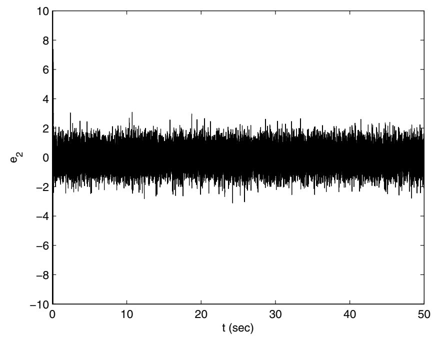

Fig. 5. An error $e_2 = x_2 - \hat{x}_2$ with fixed sampling time

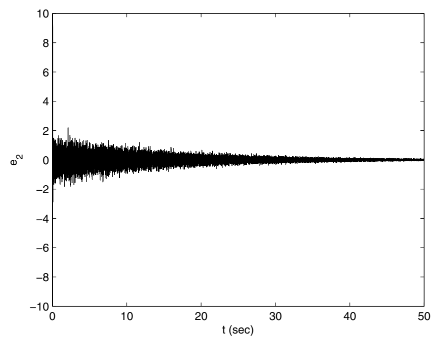

Fig. 6. An error $e_2 = x_2 - \hat{x}_2$ with SA

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