Guaranteed Performance State-Feedback Gain-Scheduling Control With Uncertain Scheduling Parameters

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State-feedback gain-scheduling controller synthesis with guaranteed performance is considered in this brief. Practical assumption has been considered in the sense that scheduling parameters are assumed to be uncertain. The contribution of this paper is the characterization of the control synthesis that parameterized linear matrix inequalities (PLMIs) to synthesize robust gain-scheduling controllers. Additive uncertainty model has been used to model measurement noise of the scheduling parameters. The resulting controllers not only ensure robustness against scheduling parameters uncertainties but also guarantee closed-loop performance in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances as well. Numerical examples and simulations are presented to illustrate the effectiveness of the synthesized controller. Compared to other control design methods from literature, the synthesized controllers achieve less conservative results as measurement noise increases.
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1 Introduction

In the vast majority of the existing works in gain-scheduling control community, it is implicitly assumed that exact measurement of scheduling parameters is available in real-time. Since uncertainties in scheduling parameters are unavoidable, perfect measurement is impossible to obtain in practical control applications. Several articles in literature dealt with the synthesis of stabilizing controllers for linear parameter varying (LPV) systems but only few techniques have been reported to cope with uncertainties associated with scheduling parameters. A gain-scheduling design method proposed in Ref. [1] to handle uncertainties in scheduling parameters. However, their method cannot cope with additive errors since the uncertainty was assumed to be proportional to the true scheduling parameters. Moreover, only the dynamic matrix was assumed to be affected by the time-varying parameters. Then, quadratic stability approach was used to address this control problem in Ref. [2]. However, it is well-known that such approach is extremely conservative and certain systems are not even quadratically stabilizable [3]. As a remedy to quadratic stability approach, parameter-dependent Lyapunov approach is used in Refs. [4,5]. However, some of the system matrices are restricted to be constant in Ref. [5] for controller synthesis. In Ref. [6], new approach is developed to handle uncertainties in scheduling parameters for filter design problem. Similar approach is used in Ref. [7] to synthesize gain-scheduling control via two stages design method; where in the first stage, stabilizing (state-feedback) controller is designed, and then the resulting controller is used in the second stage to synthesize output-feedback controller. PLMIs synthesis conditions for $\mathscr{H}_2$ dynamic output-feedback controller have been developed in Ref. [8].

The main contribution of this paper is the characterization of the control synthesis PLMIs to synthesize robust gain-scheduling controllers when scheduling parameters are not exact for state-feedback case. In other words, PLMIs conditions are derived to synthesize robust gain-scheduling controller with guaranteed closed-loop performance in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms[2]. Affine LPV systems are studied in this brief. Multisimplex modeling approach [9] has been utilized to model the scheduling parameters and their uncertainties. Matrix coefficient check relaxation approach [10] is used to relax the PLMIs conditions. To overcome conservativeness associated with quadratic stability approach, parameter-dependent Lyapunov function approach has been studied to assess stability and improve performance measures. Slack variable (SV) approach has been used to introduce additional optimization variables that decouple Lyapunov matrix from system matrices; thus, the controller can be synthesized independently from Lyapunov matrix. Following some notions that exist in the literature [11], line search with a scalar parameter has been used as an extra degree-of-freedom to improve controller performance. Compared to other design methods from literature, the synthesized controllers achieve very competitive results.

This paper is organized as follows: Problem formulation of the uncertain scheduling parameters is given in Sec. 2. Section 3 presents the modeling approach. Controller synthesis PLMIs conditions are given in Sec. 4. Section 5 presents numerical examples, simulations, and comparisons with other methods from literature. Finally, conclusions are given in Sec. 6.

2 Problem Formulation

Consider the following LPV system:

$$\begin{aligned}\dot{x}(t) &= A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) &= C(\theta(t))x(t) + D_u(\theta(t))u(t) + D_w(\theta(t))w(t)\end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^r$ is the disturbance input, $z(t) \in \mathbb{R}^p$ is the controlled output with the following matrices $A(\theta(t)) \in \mathbb{R}^{n \times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n \times m}$, $B_w(\theta(t)) \in \mathbb{R}^{n \times r}$, $C(\theta(t)) \in \mathbb{R}^{p \times n}$, $D_u(\theta(t)) \in \mathbb{R}^{p \times m}$, and $D_w(\theta(t)) \in \mathbb{R}^{p \times r}$. The system matrices in Eq. (1) are assumed to be affine parameter-dependent, i.e., each of these matrices can be represented by the following parametrization:

$$A(\theta(t)) = A_0 + \sum_{i=1}^{q} \theta_i(t) A_i \quad (2)$$

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[2]To be precise, since we are dealing with LPV systems, the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance is not well defined in the addressed problem yet. However, we use $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norm here with slightly abused terminology so that the reader can easily grasp our problem setting. We will postpone the strict definition of the control problem until the end of Sec. 3 since necessary definitions and transformations need to be introduced in Sec. 2.

The scheduling parameter vector, defined as

$$\theta(t) = [\theta_1(t), \theta_2(t), ..., \theta_q(t)]'$$

is assumed to be inexactly measured or corrupted with noise denoted by $\tilde{\theta}(t)$, such that

$$\tilde{\theta}_i(t) = (\theta_i(t) + \delta_i(t)), \quad i = 1, 2, ..., q \qquad (3)$$

where $\delta_i(t)$ represents uncertainty in the scheduling parameters and $\theta_i(t)$ is the actual value. These parameters and their uncertainties assumed to have the following known bounds:

$$-\bar{\theta}_i \leq \theta_i(t) \leq \bar{\theta}_i, -\bar{\delta}_i \leq \delta_i(t) \leq \bar{\delta}_i, \quad i = 1, 2, ..., q \qquad (4)$$

Furthermore, these parameters have their rates of variations bounded by

$$-b_{\theta_i} \leq \dot{\theta}_i(t) \leq b_{\theta_i}, \quad -b_{\delta_i} \leq \dot{\delta}_i(t) \leq b_{\delta_i}, \quad i = 1, 2, ..., q \qquad (5)$$

where each scheduling parameter and the associated uncertainty have a different rate of variation. Without loss of generality, these bounds have been assumed to be symmetric.

The goal is to synthesize gain-scheduling state-feedback controller of the form

$$u(t) = K(\tilde{\theta}(t))x(t) \qquad (6)$$

where $\tilde{\theta}(t) = \theta(t) + \delta(t)$ that stabilizes the closed-loop system while minimizing the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance costs from the disturbance input $w(t)$ to the controlled output $z(t)$, where

$$\tilde{\theta}(t) = [\tilde{\theta}_1(t), \tilde{\theta}_2(t), ..., \tilde{\theta}_q(t)],$$
$$\delta(t) = [\delta_1(t), \delta_2(t), ..., \delta_q(t)]$$

Furthermore, the controller should be robust against the uncertainties associated with the scheduling parameters as well. By substituting Eq. (6) in Eq. (1), we obtain the following closed-loop system:

$$\dot{x}(t) = A_{cl}(\theta(t), \tilde{\theta}(t))x(t) + B_w(\theta(t))w(t),$$
$$z(t) = C_{cl}(\theta(t), \tilde{\theta}(t))x(t) + D_w(\theta(t))w(t) \qquad (7)$$

where[3]

$$A_{cl}(\theta(t), \tilde{\theta}(t)) = A(\theta(t)) + B_u(\theta(t))K(\tilde{\theta}(t)),$$
$$C_{cl}(\theta(t), \tilde{\theta}(t)) = C(\theta(t)) + D_u(\theta(t))K(\tilde{\theta}(t)) \qquad (8)$$

Some terminologies and definitions are necessary to introduce now, since it will be used in Secs. 3–5.

DEFINITION 1. *Unit-simplex [9]: a unit-simplex is defined as follows*:

$$\Lambda_\ell = \left\{ \alpha \in \mathbb{R}^\ell : \sum_{i=1}^{\ell} \alpha_i = 1, \, \alpha_i \geq 0, \, i = 1, 2, ..., \ell \right\}$$

where the variable α_i varies in the unit-simplex Λ_ℓ that have ℓ vertices.

DEFINITION 2. *Multisimplex [12]: a multisimplex Λ is the Cartesian product of a finite number of q simplexes that*

$$\Lambda_{N_1} \times \Lambda_{N_2} \times \cdots \times \Lambda_{N_q} = \prod_{i=1}^{q} \Lambda_{N_i} \triangleq \Lambda$$

The dimension of the multisimplex Λ is defined as the index $N = (N_1, N_2, \cdots, N_q)$ and for simplicity of notation, $\mathbb{R}^N$ denotes for the space $\mathbb{R}^{N_1 + N_2 + \cdots + N_q}$. Thus, any variable α in the multisimplex domain Λ can be decomposed as $(\alpha_1, \alpha_2, ..., \alpha_q)$, and each α_i,

belonging into a unit-simplex Λ_{N_i}, can be decomposed as $(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{iN_i})$ for $i = 1, 2, ..., q$.

3 The Modeling Approach

In this section, systematic approach will be given to convert the scheduling parameters and the uncertainties from their original parameter space into multisimplex domain. Most of the notations and steps presented in this section are borrowed from Ref. [6].

3.1 Affine to Multisimplex Transformation. The objective of this change of variables is to construct a new convex parameter space to deal with the uncertainties in the scheduling parameters. Since we have δ_i associated with each θ_i that needs to be modeled in convex domain, we have two unit-simplexes, one for the actual scheduling parameter and the other one for its uncertainty. Note that since all time-varying parameters are assumed to be bounded above and below, all resulting unit-simplexes have two vertices, i.e., Λ_2. Thus, each of the varying parameters (θ_i and δ_i) will be modeled independently in its own simplex as follows [6]:

(1) Actual scheduling parameters ($\theta_i \Rightarrow \alpha_i$)

$$\alpha_{i1} = \frac{\theta_i + \bar{\theta}_i}{2\bar{\theta}_i} \Rightarrow \theta_i = 2\bar{\theta}_i\alpha_{i1} - \bar{\theta}_i \qquad (9)$$

$$\alpha_{i2}(t) = 1 - \alpha_{i1}(t) = 1 - \frac{\theta_i(t) + \bar{\theta}_i}{2\bar{\theta}_i} = \frac{\bar{\theta}_i - \theta_i(t)}{2\bar{\theta}_i}$$

$$\alpha_i = (\alpha_{i1}, \alpha_{i2}) \in \Lambda_2, \quad \forall i = 1, 2, ..., q$$

(2) Uncertainties ($\delta_i \Rightarrow \hat{\alpha}_i$)

$$\hat{\alpha}_{i1} = \frac{\delta_i + \bar{\delta}_i}{2\bar{\delta}_i} \Rightarrow \delta_i = 2\bar{\delta}_i\hat{\alpha}_{i1} - \bar{\delta}_i \qquad (10)$$

$$\hat{\alpha}_{i2}(t) = 1 - \hat{\alpha}_{i1}(t) = 1 - \frac{\delta_i(t) + \bar{\delta}_i}{2\bar{\delta}_i} = \frac{\bar{\delta}_i - \delta_i(t)}{2\bar{\delta}_i}$$

$$\hat{\alpha}_i = (\hat{\alpha}_{i1}, \hat{\alpha}_{i2}) \in \Lambda_2, \quad \forall i = 1, 2, ..., q$$

Thus, using this change of variables, the original affine parameter-dependent system (1) as well as the gain-scheduling controller (6) can be expressed in terms of a multisimplex variables that blend the time-varying parameters and the uncertainties together in a convex domain. Therefore, the multisimplex variables $\tilde{\alpha}$ can be defined as $\tilde{\alpha} = (\alpha, \hat{\alpha})$, with $\tilde{\alpha} \in \Lambda = \Lambda_2 \times \Lambda_2 \times \ldots \times \Lambda_2$.

Consider the case where $q = 1$ (one scheduling variable) for instance, $\alpha_1 = (\alpha_{11}, \alpha_{12})$ and $\hat{\alpha}_1 = (\hat{\alpha}_{11}, \hat{\alpha}_{12})$, then the homogeneous terms in the multisimplex variables can be written in terms of these variables as $\tilde{\alpha} = (\alpha_{11}, \alpha_{12}, \hat{\alpha}_{11}, \hat{\alpha}_{12})$.

Let $F(\tilde{\theta})$ represents the controller matrix in Eq. (6) (or any of the optimization matrices that will be given shortly in Theorems 5 and 6) that have affine dependence on the measured parameter $\tilde{\theta}$ as

$$F(\tilde{\theta}) = F_0 + \tilde{\theta}_1 F_1 = F_0 + (\theta_1 + \delta_1)F_1 \qquad (11)$$

Note that the controller is implemented in real-time using the structure of Eq. (11) that requires only sensor measurement of $\tilde{\theta}$. Substituting for θ_1 and δ_1 from Eqs. (9) and (10) yields

$$F(\tilde{\theta}) = F_0 + (2\bar{\theta}_1\alpha_{11} - \bar{\theta}_1 + 2\bar{\delta}_1\hat{\alpha}_{11} - \bar{\delta}_1)F_1 = F(\tilde{\alpha}) \qquad (12)$$

as a result, $F(\tilde{\alpha})$ is a parameter-dependent matrix that depends on time-varying parameters inside the multisimplex domain Λ. Then, applying homogenization procedure that developed in Ref. [9] to obtain

$$F(\tilde{\alpha}) = \alpha_{11}\hat{\alpha}_{11}F_{1,1} + \alpha_{11}\hat{\alpha}_{12}F_{1,2} + \alpha_{21}\hat{\alpha}_{11}F_{2,1} + \alpha_{21}\hat{\alpha}_{21}F_{2,2} \qquad (13)$$

---

[3]The dependency on t will be omitted for notational simplicity.

where the matrices $F_{1,1}$, $F_{1,2}$, $F_{2,1}$, and $F_{2,2}$ can be generated as

$$
\begin{aligned}
F_{1,1} &= F_0 + (\bar{\theta}_1 + \bar{\delta}_1)F_1, \\
F_{1,2} &= F_0 + (\bar{\theta}_1 - \bar{\delta}_1)F_1, \\
F_{2,1} &= F_0 + (-\bar{\theta}_1 + \bar{\delta}_1)F_1, \\
F_{2,2} &= F_0 + (-\bar{\theta}_1 - \bar{\delta}_1)F_1
\end{aligned}
\tag{14}
$$

This procedure can be systematically extended to handle all system matrices and optimization variables to be dependent on the multisimplex parameters $\tilde{\alpha} = (\alpha, \hat{\alpha})$ for any number of scheduling variables $q \geq 1$. The matrices F_{j_i, k_i} in Eq. (14) for $j_i = 1, 2$ and $k_i = 1, 2$ can be generated systematically as

$$
F_{j_i, k_i} = F_0 + \sum_{i=1}^{q} \left\{ (-1)^{j_i+1}\bar{\theta}_i + (-1)^{k_i+1}\bar{\delta}_i \right\} K_i
\tag{15}
$$

Remark 1. The system matrices in Eq. (1) depend only on the true scheduling parameters $\theta_i(t)$. However, the same procedure described above could be used to convert them into multisimplex domain by imposing $\bar{\delta}_i = 0$ in Eq. (15) to obtain $A(\tilde{\alpha})$, $B_w(\tilde{\alpha})$, $B_u(\tilde{\alpha})$, $C(\tilde{\alpha})$, $D_w(\tilde{\alpha})$, and $D_u(\tilde{\alpha})$.

3.2 Rate of Variation Modeling. The rates of change of each parameter and uncertainty are assumed to be bounded as in Eq. (5) for all $t \geq 0$. Since each varying parameter belongs to a unit-simplex, it is clear that the following relation should be satisfied:

$$
\dot{\alpha}_{i1}(t) + \dot{\alpha}_{i2}(t) = 0 \qquad i = 1, 2, \ldots, q
\tag{16}
$$

The relationship between the bounds of the rates of variations of scheduling parameters and the rates of changes of multisimplex variables can be obtained using Eqs. (5) and (9) to be

$$
\frac{-b_{\theta_i}}{2\bar{\theta}_i} \leq \dot{\alpha}_{i1} \leq \frac{b_{\theta_i}}{2\bar{\theta}_i}
$$

with $\dot{\alpha}_{i2} = -\dot{\alpha}_{i1}$ as the consequence of Eq. (16). Therefore, the transformation of the rate of the variations to the multisimplex domain is exact. The derivative of $\hat{\alpha}_i$ can be modeled via the same procedure using the bounds on the uncertainties $\bar{\delta}_i$.

Since $\alpha_i(t) \in \Lambda_2$ and $\hat{\alpha}_i(t) \in \Lambda_2$, the time derivatives of the parameters α_i can assume values that modeled by a convex polytope Ω_i [13]. Given the bounds b_{θ_i} and b_{δ_i} in Eq. (5), the matrices H_i (of size 2×2) can be constructed such that Ω_i is defined as

$$
\Omega_i = \left\{ \phi \in \mathbb{R}^2 : \phi = \sum_{k=1}^{2} \eta_{ik} H_i^{(k)}, \eta_i \in \Lambda_2 \right\} \\
i = 1, 2, \ldots, 2q
\tag{17}
$$

where $H_i^{(k)}$ represents the kth column of matrix H_i. Notice that, due to Eq. (16), the sum of the elements of each column of $H_i^{(k)}$ is zero. Therefore,

$$
\dot{\tilde{\alpha}} \in \Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_{2q} = \prod_{i=1}^{2q} \Omega_i
\tag{18}
$$

At this point, we are ready to define our control problem precisely.

Problem 3. Suppose that $D_w(\theta(t)) = 0$ in Eq. (1). For a given positive scalar ν, find a state-feedback controller in the form of Eq. (6) for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$ that stabilizes the closed-loop system (7) and satisfies

$$
\sup_{(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega} \mathbf{E}\left\{ \int_0^T z(t)'z(t)dt \right\} < \nu^2
\tag{19}
$$

for the disturbance input $w(t)$ given by

$$
w(t) = \begin{cases} w_0 & (t = 0) \\ \mathbf{0} & (t \neq 0) \end{cases}
$$

where w_0 is a random variable with $\mathbf{E}\{w_0 w_0'\} = I_r$ and $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation.

Problem 4. For a given positive scalar γ_∞, find a state-feedback controller in the form of Eq. (6) for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$ that stabilizes the closed-loop system (7) and satisfies

$$
\sup_{(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega} \sup_{w \in \mathcal{L}_2, w \neq 0} \frac{\|z\|_2}{\|w\|_2} < \gamma_\infty
\tag{20}
$$

The next two Lemmas characterize the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance indices, respectively.

LEMMA 1. *If there exist a continuously differentiable symmetric positive definite matrix $P(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$ and parameter-dependent matrix $W(\tilde{\alpha}) = W(\tilde{\alpha})' \in \mathbb{R}^{p \times p}$ for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$ such that the following PLMIs are satisfied [14]:*

$$
\begin{bmatrix} A_{cl}(\tilde{\alpha})P(\tilde{\alpha}) + P(\tilde{\alpha})A_{cl}(\tilde{\alpha})' - \dfrac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}}\dot{\tilde{\alpha}} & \star \\ B_w(\tilde{\alpha})' & -I_r \end{bmatrix} < \mathbf{0}
\tag{21}
$$

$$
\begin{bmatrix} P(\tilde{\alpha}) & \star \\ C_{cl}(\tilde{\alpha})P(\tilde{\alpha}) & W(\tilde{\alpha}) \end{bmatrix} > \mathbf{0}
\tag{22}
$$

$$
\mathrm{trace}(W(\tilde{\alpha})) < \nu^2
\tag{23}
$$

then the closed-loop system (7) is exponentially stable with $\mathscr{H}_2$ performance bound ν satisfies Eq. (19).

LEMMA 2. *If there exists a continuously differentiable symmetric positive definite matrix $P(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$ for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$ such that the following PLMI satisfied [15]:*

$$
\begin{bmatrix} A_{cl}(\tilde{\alpha})P(\tilde{\alpha}) + P(\tilde{\alpha})A_{cl}(\tilde{\alpha})' - \dfrac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}}\dot{\tilde{\alpha}} & \star & \star \\ C_{cl}(\tilde{\alpha})P(\tilde{\alpha}) & -I_p & \star \\ B_w(\tilde{\alpha})' & D_w(\tilde{\alpha})' & -\gamma_\infty^2 I_r \end{bmatrix} < \mathbf{0}
\tag{24}
$$

then the closed-loop system (7) is exponentially stable with $\mathscr{H}_\infty$ performance bound γ_∞ satisfies Eq. (20).

Since all scheduling parameters and their uncertainties (with their rates of variations) have been modeled as multisimplex variables, we are ready now to derive PLMIs conditions for controller synthesis which is the main contribution of this brief.

4 Gain-Scheduling Controller Synthesis

In this section, we derive controller synthesis conditions in terms of PLMIs. Note that, regarding $\mathscr{H}_2$ controller synthesis, it is assumed that $D_w(\tilde{\alpha}) = 0$ in Eq. (1) so that finite $\mathscr{H}_2$ cost can be obtained.

THEOREM 5. *For a given scalar ν, if there exist a continuously differentiable matrix $0 < P(\tilde{\alpha}) = P(\tilde{\alpha})' \in \mathbb{R}^{n \times n}$, matrices $W(\tilde{\alpha}) = W(\tilde{\alpha})' \in \mathbb{R}^{p \times p}$, $Z(\tilde{\alpha}) \in \mathbb{R}^{m \times n}$, $G(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$, and a scalar $\varepsilon > 0$, such that the following PLMIs satisfied:*

$$
\begin{bmatrix} \Phi_1(\tilde{\alpha}, \dot{\tilde{\alpha}}) & \star & \star \\ \Phi_2(\tilde{\alpha}) & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})') & \star \\ B_w(\tilde{\alpha})' & \mathbf{0}_{r \times n} & -I_r \end{bmatrix} < \mathbf{0}_{2n+r}
\tag{25}
$$

$$
\Phi_1(\tilde{\alpha}, \dot{\tilde{\alpha}}) := A(\tilde{\alpha})G(\tilde{\alpha}) + B_u(\tilde{\alpha})Z(\tilde{\alpha}) + G(\tilde{\alpha})'A(\tilde{\alpha})' \\
+ Z(\tilde{\alpha})'B_u(\tilde{\alpha})' + \frac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}}\dot{\tilde{\alpha}}
\tag{26}
$$

$$
\Phi_2(\tilde{\alpha}) := P(\tilde{\alpha}) - G(\tilde{\alpha}) + \varepsilon(A(\tilde{\alpha})G(\tilde{\alpha}) + B_u(\tilde{\alpha})Z(\tilde{\alpha}))'
\tag{27}
$$

$$\begin{bmatrix} G(\tilde{\alpha}) + G(\tilde{\alpha})' - P(\tilde{\alpha}) & \star \\ C(\tilde{\alpha})G(\tilde{\alpha}) + D_u(\tilde{\alpha})Z(\tilde{\alpha}) & W(\tilde{\alpha}) \end{bmatrix} > \mathbf{0}_{n+p} \quad (28)$$

$$\text{trace}(W(\tilde{\alpha})) < \nu^2 \quad (29)$$

then the gain-scheduling controller

$$K(\tilde{\alpha}) = Z(\tilde{\alpha})G(\tilde{\alpha})^{-1} \quad (30)$$

stabilizes the closed-loop system with $\mathscr{H}_2$ performance bound ν for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$.

Proof. Using Eq. (17), the time-derivative of Lyapunov matrix in Eq. (21) can be expressed as

$$\frac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}} \dot{\tilde{\alpha}} = \sum_{i=1}^{2q} \sum_{j=1}^{2} \frac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}_{ij}} \dot{\tilde{\alpha}}_{ij} = \sum_{i=1}^{2q} \sum_{j=1}^{2} \frac{\partial P(\tilde{\alpha})}{\partial \tilde{\alpha}_{ij}} \times (\eta_{i1}H_i(j,1)$$
$$+ \eta_{i2}H_i(j,2)) := \Pi(\tilde{\alpha}, \eta), \quad \eta_i \in \Lambda_2 \quad (31)$$

Additional SV $X(\tilde{\alpha})$ can be introduced using Finsler's Lemma [16] to decouple the dynamic matrix $A_{cl}(\tilde{\alpha})$ from Lyapunov matrix $P(\tilde{\alpha})$, hence

$$\Psi(\tilde{\alpha}) + X(\tilde{\alpha})V(\tilde{\alpha}) + V(\tilde{\alpha})'X(\tilde{\alpha})' < 0 \quad (32)$$

where

$$\Psi(\tilde{\alpha}) \triangleq \begin{bmatrix} \Pi(\tilde{\alpha}, \eta) & P(\tilde{\alpha}) & 0 \\ P(\tilde{\alpha}) & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \quad (33)$$

$$X(\tilde{\alpha}) \triangleq \begin{bmatrix} G(\tilde{\alpha})' & 0 \\ R(\tilde{\alpha})' & 0 \\ 0 & I \end{bmatrix}, \quad V(\tilde{\alpha}) \triangleq \begin{bmatrix} A_{cl}(\tilde{\alpha})' & -I & 0 \\ B_w(\tilde{\alpha})' & 0 & -I \end{bmatrix} \quad (34)$$

such that $V(\tilde{\alpha})^{\perp'} \Psi(\tilde{\alpha}) V(\tilde{\alpha})^{\perp} < 0$, with $V(\tilde{\alpha})^{\perp'} = [I \quad A_{cl}(\tilde{\alpha}) \quad B_w(\tilde{\alpha})]$. Therefore, substituting Eqs. (33) and (34) into Eq. (32) to obtain

$$\begin{bmatrix} \Pi(\tilde{\alpha}, \eta) & P(\tilde{\alpha}) & 0 \\ P(\tilde{\alpha}) & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})' & 0 \\ R(\tilde{\alpha})' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}(\tilde{\alpha})' & -I & 0 \\ B_w(\tilde{\alpha})' & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} A_{cl}(\tilde{\alpha}) & B_w(\tilde{\alpha}) \\ -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} G(\tilde{\alpha}) & R(\tilde{\alpha}) & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \quad (35)$$

At this point, it is important to impose particular structure to the SV $X(\alpha)$ to provide convex parametrization. Therefore, choosing $R(\tilde{\alpha}) = \varepsilon G(\tilde{\alpha})$ is sufficient to keep convexity of Eq. (35), where ε is a scalar used to provide extra degree-of-freedom to perform line search to reduce conservatism. Using the change of variable $Z(\tilde{\alpha}) = K(\tilde{\alpha})G(\tilde{\alpha})$ yields

$$\begin{bmatrix} \Pi(\tilde{\alpha}, \eta) & P(\tilde{\alpha}) & 0 \\ P(\tilde{\alpha}) & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})'A_{cl}(\tilde{\alpha})' & -G(\tilde{\alpha})' & 0 \\ \varepsilon G(\tilde{\alpha})'A_{cl}(\tilde{\alpha})' & -\varepsilon G(\tilde{\alpha})' & 0 \\ B_w(\tilde{\alpha})' & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} A_{cl}(\tilde{\alpha})G(\tilde{\alpha}) & \varepsilon A_{cl}(\tilde{\alpha})G(\tilde{\alpha}) & B_w(\tilde{\alpha}) \\ -G(\tilde{\alpha}) & -\varepsilon G(\tilde{\alpha}) & 0 \\ 0 & 0 & -I \end{bmatrix} < 0$$

that directly leads to Eq. (25). Multiplying Eq. (28) from left by $[C_{cl}(\tilde{\alpha}) \quad -I]$ and by its transpose from right with $C_{cl}(\tilde{\alpha}) = C(\tilde{\alpha}) + D_u(\tilde{\alpha})K(\tilde{\alpha})$ to obtain $W(\tilde{\alpha}) > C_{cl}(\tilde{\alpha})P(\tilde{\alpha})$ $C_{cl}(\tilde{\alpha})'$, with Schur complement, Eq. (22) can be recovered. The LMI (29) ensures that ν is the guaranteed bound of the $\mathscr{H}_2$ norm of the closed-loop system.

THEOREM 6. *For a given scalar γ_∞, if there exist a continuously differentiable matrix $0 < P(\tilde{\alpha}) = P(\tilde{\alpha})' \in \mathbb{R}^{n \times n}$, matrices*

$Z(\tilde{\alpha}) \in \mathbb{R}^{m \times n}$, $G(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$, *and a scalar $\varepsilon > 0$, such that the following PLMI satisfied:*

$$\begin{bmatrix} \Phi_1(\tilde{\alpha}, \dot{\tilde{\alpha}}) & * & * & * \\ \Phi_2(\tilde{\alpha}) & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})') & * & * \\ \Phi_3(\tilde{\alpha}) & \Phi_4(\tilde{\alpha}) & -I_p & * \\ B_w(\tilde{\alpha})' & \mathbf{0}_{r \times n} & D_w(\tilde{\alpha})' & -\gamma_\infty^2 I_r \end{bmatrix} < \mathbf{0} \quad (36)$$

with $\Phi_1(\tilde{\alpha}, \dot{\tilde{\alpha}})$ and $\Phi_2(\tilde{\alpha})$ defined in Eqs. (26) and (27), and

$$\Phi_3(\tilde{\alpha}) = C(\tilde{\alpha})G(\tilde{\alpha}) + D_u(\tilde{\alpha})Z(\tilde{\alpha}),$$
$$\Phi_4(\tilde{\alpha}) = \varepsilon C(\tilde{\alpha})G(\tilde{\alpha}) + \varepsilon D_u(\tilde{\alpha})Z(\tilde{\alpha})$$

then the gain-scheduling controller

$$K(\tilde{\alpha}) = Z(\tilde{\alpha})G(\tilde{\alpha})^{-1}$$

stabilizes the closed-loop system with a guaranteed $\mathscr{H}_\infty$ performance bound γ_∞ for any pair $(\tilde{\alpha}, \dot{\tilde{\alpha}}) \in \Lambda \times \Omega$.

Proof. Defining the projection matrix

$$T(\tilde{\alpha}) = \begin{bmatrix} I & A_{cl}(\tilde{\alpha}) & 0 & 0 \\ 0 & C_{cl}(\tilde{\alpha}) & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

with

$$A_{cl}(\tilde{\alpha}) = A(\tilde{\alpha}) + B_u(\tilde{\alpha})K(\tilde{\alpha}),$$
$$C_{cl}(\tilde{\alpha}) = C(\tilde{\alpha}) + D_u(\tilde{\alpha})K(\tilde{\alpha})$$

multiplying Eq. (36) by $T(\tilde{\alpha})$ from left and by $T(\tilde{\alpha})'$ from the right. Since $\tilde{\alpha} \in \Omega$ defined in Eq. (18), the PLMI (24) can be obtained which represents the bounded real lemma for time-varying systems; therefore, γ_∞ is an upper bound for the $\mathscr{H}_\infty$ norm of the closed-loop system.

Remark 2. Theorems 5 and 6 encompass robust (parameter-independent) controller synthesis as a special case. To illustrate, constraining the synthesis variables to be constant matrices, i.e., $G(\tilde{\alpha}) = G$ and $Z(\tilde{\alpha}) = Z$, a robust controller $K = ZG^{-1}$ can be synthesized as direct result of these two theorems.

4.1 PLMIs Relaxation. As shown above, the synthesis conditions of Theorems 5 and 6 are formulated as PLMIs (for a fixed scalar ε) in terms of time-varying parameters inside the multisimplex domain, which is a special type of convex optimization problem. PLMIs are equivalent to infinite dimensional LMI constraints and it seems that it is difficult to be solved numerically at this stage. However, modern robust optimization techniques considerably strengthened this framework by providing a rigorous ways to deal with the PLMIs [17]. Several powerful numerical and computational tools have been developed recently by many researchers independently that are aimed to approximate the PLMIs [10,17]; to mention a few: sum-of-square matrices [18], SV approach [19], and coefficient check approach using Pólya's theorem [10]. Exploiting these relaxation methods overcomes the difficulty for solving the PLMIs by converting them into finite-dimensional LMIs as demonstrated in Ref. [20].

Finally, the relaxation approach developed in Ref. [9] is adopted in this paper to relax the PLMIs conditions of Theorems 5 and 6, since it supports PLMIs that depends on multisimplex parameters. It is worth noting that the algebraic manipulation of PLMIs is a sophisticated procedure. However, a specialized parser, ROLMIP[4] [21], has been recently developed as a tool to perform such manipulation and relaxation of the PLMIs. This package works jointly with the LMI parser YALMIP [22] and the solver SeDuMi [23] that have been used in this paper to obtain the optimal solution of the convex optimization (synthesis) problem.

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[4]Available for download at http://www.dt.fee.unicamp.br/~agulhari/rolmip/rolmip.htm

5 Numerical Examples

The objective of the numerical examples and simulation results presented in this section is to demonstrate the advantage of the PLMIs conditions proposed in this paper. Comparison results only with the approach described in Ref. [4] will be included since it represents the state-of-the-art for gain-scheduling control with uncertain scheduling parameters.

Example 1. Consider the following LPV system that represents the dynamics of two-masses and two-springs [4]:

$$A(\theta(t)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -\theta_1(t) & 0 \\ 2 & -2 & 0 & -2\theta_1(t) \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_u = [0 \ \ 0 \ \ 0 \ \ 1]',$$

$$C = [0 \ \ 1 \ \ 0 \ \ 0], \quad D_w = 0, \quad D_u(\theta(t)) = \theta_2(t)$$

with the following bounds:

$$0.5 \le \theta_1(t) \le 3.5, \quad 0.5 \le \theta_2(t) \le 1.5, \quad |\dot{\theta}_q(t)| \le \kappa,$$

$$|\delta_q(t)| \le \zeta, \quad |\dot{\delta}_q(t)| \le 10 \times \zeta, \quad q = 1, 2$$

Theorem 5 is used to synthesize gain-scheduling controllers for different bounds of measurement noise. Table 1 presents the achieved $\mathscr{H}_2$ bound ν. Clearly, the $\mathscr{H}_2$ bounds are influenced by the uncertainty $\delta_q(t)$ associated with the scheduling parameters $\theta_q(t)$ for $q = 1, 2$. Table 2 from Ref. [4] is given here to facilitate comparison with our controllers. Although the achievable performance for the two methods is very close, it can be observed that the controllers synthesized via Theorem 5 outperform the method of Ref. [4] for large values of ζ and κ.

In addition to the gain-scheduling controller, robust controller (parameter-independent) has been synthesized (see Remark 2) with its performance shown in the last row of Table 1. Note that as the uncertainty size increases, the achieved $\mathscr{H}_2$ performance (for the gain-scheduling controller) deteriorates. For example, when $\zeta = 2$, the achieved performance is the same as the performance provided by the robust controller. This is a logical observation since as the uncertainty of the scheduling parameter

increases, the measurement is not reliable to be used for scheduling anymore. In this case, there is no benefit for the designer to implement gain-scheduling controller over the robust one since the achievable performance is the same.

Example 2. Consider the following LPV system [4]:

$$A(\theta(t)) = \begin{bmatrix} 25.9 - 60\theta(t) & 1 \\ 20 - 40\theta(t) & 34 - 64\theta(t) \end{bmatrix}, \quad B_u = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$$B_w = \begin{bmatrix} -0.03 \\ -0.47 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The varying parameter $\theta(t)$ has the following bounds $0 \le \theta(t) \le 1$, $|\dot{\theta}(t)| \le \kappa$, with measurement uncertainty bound $|\delta(t)| \le \zeta$ and $|\dot{\delta}(t)| \le 10 \times \zeta$.
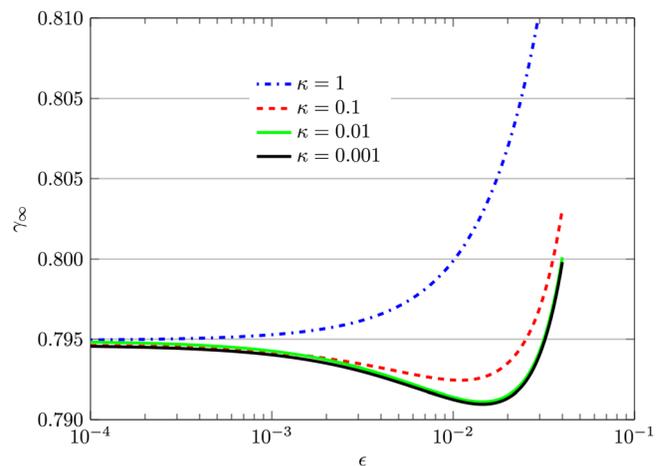
Theorem 6 is used to synthesize $\mathscr{H}_\infty$ gain-scheduling controller for this example. The guaranteed $\mathscr{H}_\infty$ performance is shown in Table 3, as expected, the performance deteriorates as δ and κ increase. Robust controller has been synthesized as well to study its performance compared to the gain-scheduling controller. The performance of the robust controller is shown in the last row of Table 3. While the gain-scheduling controller shows better performance for low range of uncertainty size, it provides no

Table 3 $\mathscr{H}_\infty$ Guaranteed cost using Theorem 6

ζ	κ			
	0.001	0.01	0.1	1
0.01	0.583	0.584	0.589	0.652
0.1	0.718	0.718	0.727	0.793
0.2	0.791	0.791	0.792	0.795
0.5	0.795	0.795	0.795	0.795
Robust	0.795	0.795	0.795	0.795

Table 4 $\mathscr{H}_\infty$ Guaranteed cost method of Ref. [4]

ζ	κ			
	0.001	0.01	0.1	1
0.01	0.480	0.489	0.594	0.795
0.1	0.612	0.622	0.732	0.795
0.2	0.752	0.763	0.795	0.795
0.5	0.795	0.795	0.795	0.795
Robust	0.795	0.795	0.795	0.795

Table 1 $\mathscr{H}_2$ Guaranteed cost using Theorem 5

ζ	κ			
	0.001	0.01	0.1	1
0.1	0.092	0.093	0.097	0.129
0.5	0.276	0.276	0.278	0.285
1	0.278	0.279	0.280	0.287
2	0.282	0.283	0.286	0.295
Robust	0.282	0.283	0.286	0.295

Table 2 $\mathscr{H}_2$ Guaranteed cost method of Ref. [4]

ζ	κ			
	0.001	0.01	0.1	1
0.1	0.070	0.077	0.139	0.147
0.5	0.258	0.263	0.299	0.300
1	0.284	0.289	0.300	0.300
2	0.300	0.300	0.300	0.300
Robust	0.300	0.300	0.300	0.300

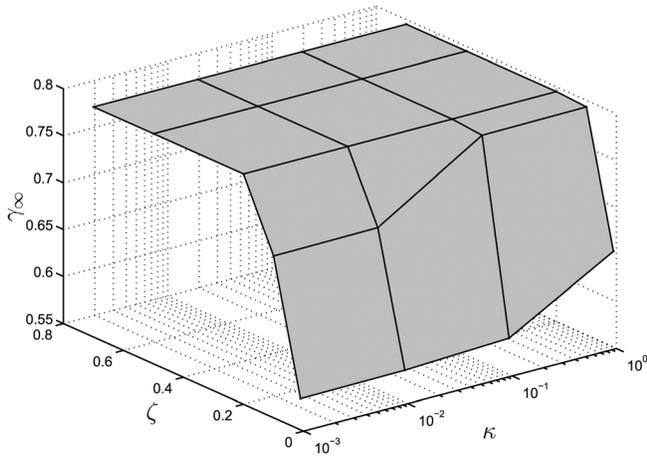**Fig. 1   Performance versus ϵ with $\zeta = 0.2$**

Fig. 2 Guaranteed performance

significant improvement over the robust controller when $\zeta \geq 0.5$. Table 4 shows the results of Ref. [4] for the same example. Comparing these two tables, it can be observed that our approach achieves very competitive results with those associated with Ref. [4]. A line search for ε with a linear-grid of 350 points between 10^{-4} and 10^{-1} has been conducted and is shown in Fig. 1 for $\zeta = 0.2$ with different rates of change (κ). Figure 2 shows the guaranteed $\mathscr{H}_\infty$ performance as function of ζ and κ.

Simulation study has been conducted for this example to illustrate robustness of the synthesized controller against the mismatch between the actual and measured scheduling parameters. A scheduling parameter that is defined as $\theta(t) = 05 + 05\sin(02t)$ and a

noisy version of this signal $(\tilde{\theta}(t))$ are both shown in Fig. 3(a). A random noise with bounds $|\delta(t)| \leq 0.075$ and $|\dot{\delta}(t)| \leq 1$ has been intentionally added to the actual scheduling parameter to imitate the measurement noise. Then, Theorem 6 is used to synthesize controller (with $\varepsilon = 0.001$) at the vertices of the multisimplex domain as

$$K_1 = [2.060 \quad -84.399], \quad K_2 = [-0.725 \quad -50.508],$$
$$K_3 = [1.096 \quad -75.031], \quad K_4 = [-0.837 \quad -48.307]$$

To simulate the closed-loop system, an $\mathscr{L}_2$ disturbance signal defined by $w(t) = \exp(-04t)$ is generated as disturbance input. The responses to this disturbance for both cases are shown in Fig. 3(b). Clearly, the noise amplitude in the response (associated with the noisy scheduling parameter) is much less than the noise amplitude in the measured scheduling parameter. This simulation result not only shows good robustness against measurement noise but also good disturbance attenuation as well.

6 Conclusions

This brief deals with state-feedback gain-scheduling controller synthesis for polytopic LPV systems with noisy scheduling parameters. PLMIs conditions have been derived to synthesize controller with a prescribed performance measure in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms. Our synthesis conditions encompass robust controller synthesis as a special case. Numerical examples and simulations are given to study the effect of uncertainty on the guaranteed performance. Compared with other approaches from literature, the proposed controllers are less conservative when noise level is high. The effectiveness of the synthesize approach is very tempting to be extended to handle dynamic output-feedback controllers in future work. In addition, experimental validation is another direction for future study to demonstrate practicality of the developed synthesis conditions.

Nomenclature

I_n = identity matrix of size $n \times n$
  n = is omitted when the size of the identity can be easily inferred from the context
$0_{n \times p}$ = zero matrices of size $n \times p$
  $\star$ = the transpose of the off-diagonal matrix block
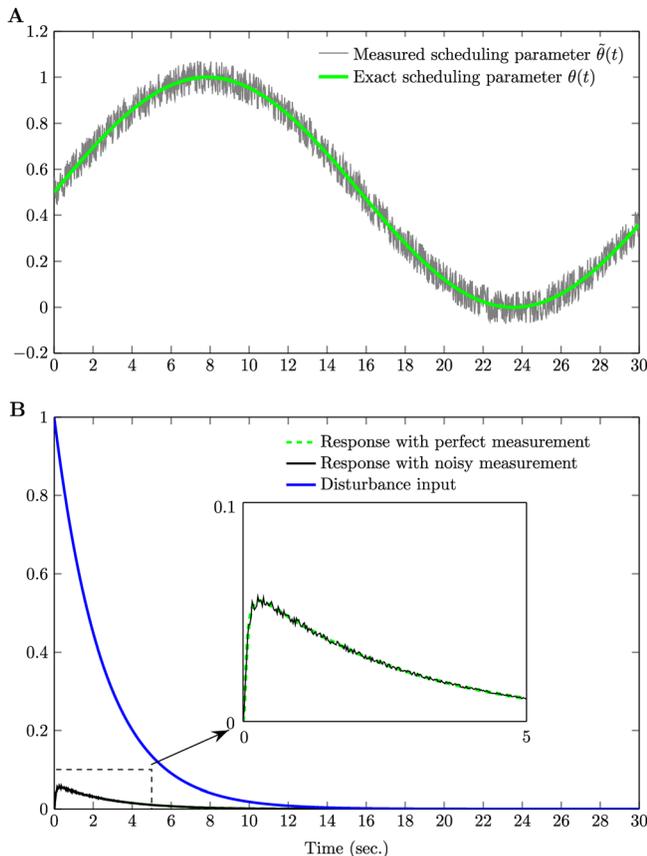

Fig. 3 Simulation: (a) measured and exact scheduling parameters and (b) disturbance attenuation responses associated with exact and noisy scheduling parameter

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