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Linear Matrix Inequalities Approach to Input Covariance Constraint Control With Application to Electronic Throttle

In this paper, the input covariance constraint (ICC) control problem is solved by convex optimization subject to linear matrix inequalities (LMIs) constraints. The ICC control problem is an optimal control problem that is concerned to obtain the best output performance subject to multiple constraints on the input covariance matrices. The contribution of this paper is the characterization of the control synthesis LMIs used to solve the ICC control problem. Both continuous- and discrete-time problems are considered. To validate our scheme in real-world systems, ICC control based on convex optimization approach was used to control the position of an electronic throttle plate. The controller performance compared experimentally with a well-tuned base-line proportional-integral-derivative (PID) controller. Comparison results showed that not only better performance has been achieved but also the required control energy for the ICC controller is lower than that of the base-line controller. [DOI: 10.1115/1.4030525]

1 Introduction

ICC control problem is an optimal control problem in which the output performance is minimized subject to multiple constraints on the control input covariance matrices U_i of the form $U_i \leq \bar{U}_i$, where $\bar{U}_i > 0$ is given. The ICC control problem has two interesting interpretations: stochastic and deterministic. For the stochastic interpretation, the exogenous inputs are assumed to be uncorrelated zero-mean white noises with a given intensity. With the exogenous input defined in this way, the ICC control problem minimizes the weighted performance output covariance subject to the control ICCs, such that the constraints can be interpreted as constraints on the variance of the control actuation. For the deterministic interpretation, the exogenous inputs are assumed to be unknown disturbances that belong to a bounded $\mathcal{L}_2$ energy set. Then, the ICC control problem minimizes the maximum singular value of the performance outputs while ensuring that the maximum singular value of the control inputs is less than the corresponding control input constraints. In other words, the ICC control problem is the problem of minimizing the weighted sum of worst-case peak values on the performance outputs subject to the constraints on the worst-case peak values of the control input. This interpretation is important in applications where hard constraints on the actuator signals are present, such as space telescope pointing control [1], navigation systems [2], structures, aerospace [3] and machine tool control.

When controlling any physical system, actuators are used to drive the system in a particular way to achieve desired output response. For practical applications, these actuators typically have a finite amount of power available to drive the system. When closed-loop controllers are designed using conventional methods, these actuator constraints are usually not taken into consideration, and thus, it is possible to design a controller that can command the actuator to supply more control energy than what is possible. When this happens, the control input saturates and leads to performance degradation. Hence, it is important to guarantee that the system remains in the linear range; otherwise, stability of the closed-loop system is no longer guaranteed. Thus, actuator constraints have to be considered in the early stage of the control synthesis. Therefore, the controller with the best possible performance is obtained with respect to a given set of available actuators for the system. Additionally, when there are multiple actuators available to control a multi-input physical system, it can be difficult to know how to obtain the best performance. The ICC control problem not only guarantees that the control will not command more than what each actuator is capable to supply, but it also synthesizes a controller for the multi-input system that obtains the best possible performance. For instance, considering a system with multiple actuators, to meeting the performance requirement a conventional design could lead to a controller that requires one of the actuators to operator beyond its physical constraint, while an ICC control design would lead to a satisfactory control without exceeding any actuator constraint by redistributing actuator utilization.

Covariance control problems witnessed rapidly growing research interest due to its applicability to different engineering applications (see Refs. [4–9]), many researchers addressed these control problems in various contexts. Hsieh et al. [10] considered the constrained linear quadratic control problem that minimizes the control energy while satisfying output covariance constraints (OCCs). Then, Zhu et al. [11] developed a convergent algorithm that solves the OCC control problem. The OCC control problem is an optimal control problem that minimizes the control input subject to OCCs [12]. The problem has been solved by a linear quadratic Gaussian (LQG) controller with a special choice of output weights, which can be obtained by using iterative OCC algorithm detailed in Ref. [11]. Considering some other available algorithms in literature, a fuzzy logic algorithm has been proposed

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to solve the variance constrained LQG control in Ref. [13]. Another algorithm based on derivative information of the relative $\mathcal{H}_2$ cost is developed to achieve quasi-Newton convergence in Ref. [14]. Although variations of these algorithms have been already considered in the literature, to the best of the authors' knowledge, none of them considers the more realistic problem with direct, convex, and noniterative nature. Therefore, the objective of the current paper is to provide synthesis conditions based on convex optimization approach without using iterative algorithms that their convergence could be not guaranteed.

The ICC control problem is closely related to the OCC control problem. While the ICC control problem can also technically be solved by LQG controller with special choice of input weights, developing an iterative algorithm to obtain such optimal (input) weighting matrix is a difficult problem to solve. However, after reconsidering the OCC control problem as a convex optimization with LMIs constraints in Ref. [15], it became clear that the ICC control problem could also be solved as a convex optimization with LMIs constraints [16]. Hence, the contribution of this paper is the characterization of state- and output-feedback control synthesis LMIs conditions required to solve the ICC control problem for both continuous- and discrete-time systems. From practical view point, constraints on actuator size (available control energy) are inevitable. With the developed ICC control strategy, various constraints on different control channels can be easily satisfied.

This paper is organized as follows: Section 2 introduces the continuous-time ICC control problem and then presents theorems 2 and 3 that provide LMIs conditions that can be solved to obtain state-feedback or dynamic output-feedback controllers, respectively, that minimize an upper bound on the ICC cost. In Sec. 3, the discrete-time ICC control problem is presented. Theorems 5 and 6 are given to provide synthesis conditions for state-feedback and full-order dynamic output-feedback controller that minimizes the upper bound of the ICC performance. To demonstrate the effectiveness of the developed controller synthesis conditions introduced in this paper, an ICC controller has been designed and implemented experimentally to control plate position of an electronic throttle system in Sec. 4. Simulations, experimental results, and comparisons of the ICC and the PID controllers are presented in Sec. 5. Concluding remarks are given in Sec. 6.

The notation used in this paper is fairly standard. The positive definiteness of a matrix A is denoted by $A > 0$. $\mathbb{R}$ denotes the set of real numbers. The symbol $\star$ is used to represents the transpose of the off-diagonal matrix block. Trace (A) denotes the trace of the matrix A, which represents the sum of diagonal elements of the matrix A. $\boldsymbol{I}$ and 0 are used to refer to identity and zero matrices, respectively. The transpose of a matrix A is referred as A'. The following notation $(\bullet)'$ is used to express $A + A'$. Other notations will be explained in due course.

2 Continuous-Time Systems

Consider the following continuous-time system:

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t) + D_p w_p(t) \\ y_p(t) &= C_p x_p(t) \\ z(t) &= M_p x_p(t) + v(t) \end{aligned} \tag{1}$$

where $x_p(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, w_p(t) \in \mathbb{R}^r$, and $v(t) \in \mathbb{R}^s$ represent the state, control, process noise, and measurement noise, respectively. The vector $y_p(t) \in \mathbb{R}^\ell$ contains all variables, whose dynamic responses are of interest and the vector $z(t) \in \mathbb{R}^q$ is a vector of noisy measurements.

Suppose that we apply to the plant (1) a full state-feedback stabilizing control law of the form

$$u(t) = K x_p(t) \tag{2}$$

or a strictly proper output-feedback stabilizing control law given by

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c z(t) \\ u(t) &= C_c x_c(t) \end{aligned} \tag{3}$$

Then, the resulting closed-loop system is

$$\begin{aligned} \dot{x}(t) &= A x(t) + D w(t) \\ y(t) &= \begin{bmatrix} y_p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C_y \\ C_u \end{bmatrix} x(t) = C x(t) \end{aligned} \tag{4}$$

where for the state-feedback case we have $x(t) = x_p(t)$ and $w(t) = w_p(t)$, while for the output-feedback case we have

$$x(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}, \quad w = \begin{bmatrix} w_p \\ v \end{bmatrix}$$

Considering the closed-loop system (4), let W_p and W_v denote positive definite symmetric matrices with dimensions equal to the process noise $w_p(t)$ and measurement vector $z(t)$, respectively. Then, define $W = W_p$, if the state-feedback controller (2) is used or

$$W = \begin{bmatrix} W_p & \mathbf{0} \\ \mathbf{0} & W_v \end{bmatrix} \tag{5}$$

If strictly proper, output-feedback controller (3) is used. Let $\bar{P}$ denote the closed-loop controllability Gramian from the (weighted) disturbance input $W^{-1/2}w$. Since A is stable, $\bar{P}$ satisfies

$$0 = A\bar{P} + \bar{P}A' + DWD' \tag{6}$$

The control input $u(t)$ in Eq. (4) is partitioned into

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

such that each $u_i(t)$ for $i = 1, 2, \ldots, m$ can be written as

$$u_i(t) = C_{u,i} x_p(t) = \Phi_i C_u x_p(t) \in \mathbb{R}$$

where Φ_i is an appropriately selected projection matrix for each input u_i. In this paper, we are interested in finding controllers of the form (2) or (3) that minimize the (weighted) output performance trace$(QC_p\bar{P}C_p')$ with $Q > 0$, and satisfy the constraints

$$U_i = \Phi_i C_u \bar{P} C_u' \Phi_i' \leq \bar{U}_i, \quad i = 1, 2, \ldots, m \tag{7}$$

where $\bar{U}_i > 0$ $(i = 1, 2, \ldots, m)$ are given constraints, $\bar{P}$ solves (6), and Q is suitably chosen output weighting matrix. This problem, which is called the ICC problem, is defined as follows.

Problem 1. *Continuous-time ICC control: find a static state-feedback or full-order dynamic output-feedback controller for the system (1) to minimize the ICC cost*

$$J_{\mathrm{ICC}} = \mathrm{trace}\left(QC_y\bar{P}C_y'\right), \quad Q > 0 \tag{8}$$

subject to Eqs. (6) and (7).

In this paper, we consider a convex optimization approach to solve the ICC control problem via LMI formulation. The next theorem formulates the problem in terms of LMIs conditions that can be easily solved using the available interior-point polynomial time methods [17].

THEOREM 1. *Consider the closed-loop system (4). Given the input constraints $\bar{U}_i$ for $i = 1, 2, \ldots, m$, if there exists a symmetric matrix P such that the following LMIs are satisfied*:

$$\begin{bmatrix} AP + PA' & DW^{1/2} \\ W^{1/2}D' & -\mathbf{I} \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} \bar{U}_i & \Phi_i C_u P \\ PC_u'\Phi_i' & P \end{bmatrix} > 0 \tag{10}$$

then the closed-loop system (4) is asymptotically stable with an input covariance bounded by

$$\bar{U}_i > \Phi_i C_u P C_u' \Phi_i' > \Phi_i C_u \bar{P} C_u' \Phi_i' = U_i, \quad \forall i = 1, 2, \ldots, m \tag{11}$$

and an ICC cost bounded by

$$\begin{aligned} \bar{J}_{\mathrm{ICC}} &= \mathrm{trace}\left(QC_y P C_y'\right) \\ &\geq \mathrm{trace}\left(QC_y \bar{P} C_y'\right) = J_{\mathrm{ICC}} \end{aligned} \tag{12}$$

Proof. According to Ref. [18], the Lyapunov equation (6) can be written as the following inequality:

$$0 > AP + PA' + DWD' \tag{13}$$

where $P = P' > 0$. Notice that Eq. (13) is the Schur complement of the LMI (9). Since $P = P' > 0$, to ensure that Eq. (9) is satisfied, the closed-loop state matrix A must be Hurwitz. Since (13) is less than zero, there exists a matrix $M = M' > 0$ such that

$$0 = AP + PA' + DWD' + M$$

Consequently, $P > \bar{P}$ yields

$$\mathrm{trace}\left(QC_y P C_y\right) > \mathrm{trace}\left(QC_y \bar{P} C_y\right)$$

such that (12) holds. To upper bound the input covariance, the following inequality must be true:

$$\bar{U}_i - \Phi_i C_u P C_u' \Phi_i' > 0, \quad \forall i = 1, 2, \ldots, m \tag{14}$$

Since the matrix C_u is made-up of controller matrices, the inequality (14) will result in multiplications between controller matrices and the Lyapunov matrix P that leads to a bilinear matrix inequality due to this multiplication. To handle this issue, we note that

$$\bar{U}_i - \Phi_i C_u P P^{-1} P C_u' \Phi_i' > 0, \quad \forall i = 1, 2, \ldots, m$$

is the Schur complement of the LMI (10), such that Eq. (11) holds.

2.1 State-Feedback Control.

With the state-feedback controller (2), the closed-loop system matrices in Eq. (4) are given by

$$\begin{aligned} A &= A_p + B_p K, \quad DW^{1/2} = D_p W_p^{1/2}, \\ C_y &= C_p, \quad C_u = K \end{aligned} \tag{15}$$

The next theorem provides synthesis conditions for state-feedback controller.

THEOREM 2. *There exists a controller in the form (2), given by*

$$K = LP^{-1}$$

that minimizes J_{ICC} (8) and satisfies the input constraints (7), if there exists a matrix $L \in \mathbb{R}^{m\times n}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ that minimize the upper bound of the ICC cost

$$\begin{aligned} \bar{J}_{\mathrm{ICC}} &= \min_{P,L} \mathrm{trace}\left(QC_p P C_p'\right) \\ &> \mathrm{trace}\left(QC_y \bar{P} C_y'\right) = J_{\mathrm{ICC}} \end{aligned}$$

subject to the LMIs

$$\begin{bmatrix} A_p P + B_p L + (\bullet)' & D_p W_p^{1/2} \\ \star & -\mathbf{I} \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} \bar{U}_i & \Phi_i L \\ \star & P \end{bmatrix} > 0 \tag{17}$$

for $i = 1, 2, \ldots, m$.

Proof. Substituting the closed-loop matrices (15) into the LMI (9) of theorem 1 results in the following bilinear matrix inequality:

$$\begin{bmatrix} \left(A_p + B_p K\right)P + (\bullet)' & D_p W_p^{1/2} \\ W_p^{1/2} D_p' & -\mathbf{I} \end{bmatrix} < 0$$

Multiplying the Lyapunov matrix P into the parenthesis gives

$$\begin{bmatrix} A_p P + B_p K P + (\bullet)' & D_p W_p^{1/2} \\ W_p^{1/2} D_p' & -\mathbf{I} \end{bmatrix} < 0$$

which clearly shows multiplication between the state-feedback controller K and the Lyapunov matrix P. However, by using the change of variables $L = KP$ [19], the LMI (16) is constructed. The same procedure is used to construct the LMI (17) as well. As a result of applying theorem 1, when the LMIs (16) and (17) are satisfied, the closed-loop system formed with the state-feedback controller K makes the closed-loop system asymptotically stable while minimizing the upper bound of the ICC cost $\bar{J}_{\mathrm{ICC}}$ and satisfying the ICCs (7).

2.2 Dynamic Output-Feedback Control.

This subsection provides synthesis conditions for strictly proper, full-order dynamic output-feedback controller defined in Eq. (3), the closed-loop matrices in Eq. (4) are given by

$$\begin{aligned} A &= \begin{bmatrix} A_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix}, \quad DW^{1/2} = \begin{bmatrix} B_w \\ B_c D_{zw} \end{bmatrix} \\ C_y &= \begin{bmatrix} C_p & 0 \end{bmatrix}, \quad C_u = \begin{bmatrix} 0 & C_c \end{bmatrix} \end{aligned} \tag{18}$$

where $B_w = \begin{bmatrix} D_p W_p^{1/2} & \mathbf{0} \end{bmatrix}$ and $D_{zw} = \begin{bmatrix} \mathbf{0} & W_v^{1/2} \end{bmatrix}$.

THEOREM 3. *There exists a controller in the form (3), given by*

$$\begin{aligned} A_c &= V^{-1}\left(R - YA_p X - YB_p L - FM_p X\right)U^{-1} \\ B_c &= V^{-1}F \\ C_c &= LU^{-1} \end{aligned}$$

that minimizes J_{ICC} (8) and satisfies the input constraints (7), if there exist matrices $L \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{n\times q}$, and $R \in \mathbb{R}^{n\times n}$ and symmetric matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ that minimize the upper bound of the ICC cost

$$\begin{aligned} \bar{J}_{\mathrm{ICC}} &= \min_{X,Y,R,L,F} \mathrm{trace}\left(QC_p X C_p'\right) \\ &> \mathrm{trace}\left(QC_y \bar{P} C_y'\right) = J_{\mathrm{ICC}} \end{aligned}$$

subject to the LMIs

$$\begin{bmatrix} A_pX + B_pL + (\bullet)' & A_p + R' & B_w \\ \star & YA_p + FM_p + (\bullet)' & YB_w + FD_{zw} \\ \star & \star & -\mathbf{I} \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} \bar{U}_i & \Phi_i L & 0 \\ \star & X & \mathbf{I} \\ \star & \star & Y \end{bmatrix} > 0 \tag{20}$$

for $i = 1, 2, \ldots, m$.

Proof. Substituting the closed-loop matrices (18) into the LMIs conditions of theorem 1 results in nonlinear matrix inequalities. To obtain a set of LMIs conditions, a nonlinear transformation (change-of-variables) is performed. First, the Lyapunov matrix P, its inverse P^{-1} and the controller matrix $\mathscr{K}$ are partitioned as

$$P := \begin{bmatrix} X & U' \\ U & Z_1 \end{bmatrix}, \quad P^{-1} := \begin{bmatrix} Y & V \\ V' & Z_2 \end{bmatrix}, \quad \mathscr{K} := \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$$

Then, the transformation matrix

$$\mathcal{J} := \begin{bmatrix} I & Y \\ \mathbf{0} & V' \end{bmatrix}$$

that is used in Refs. [19–22] introduced, and the following nonlinear change of variables is defined:

$$\begin{bmatrix} R & F \\ L & 0 \end{bmatrix} := \begin{bmatrix} V & YB_p \\ 0 & I \end{bmatrix} \mathscr{K} \begin{bmatrix} U & 0 \\ M_pX & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A_p [X \quad 0]$$

After substituting the closed-loop matrices (18) into the LMIs conditions of theorem 1, the following congruence transformations:

$$T_1 = \mathrm{diag}(\mathcal{J}, I), \quad T_2 = \mathrm{diag}(I, \mathcal{J})$$

will be applied to the LMIs (9) and (10) to obtain the LMIs (19) and (20), respectively. For example, the LMI (19) is obtained by multiplying the LMI (9) by T_1 from the right and T_1' from the left. Then, as a result of applying Theorem 1, when the LMIs (19) and (20) are satisfied, the closed-loop system formed with the strictly proper, output-feedback controller (3) ensures asymptotic stability while minimizing the upper bound of the ICC cost $\bar{J}_{\mathrm{ICC}}$ and satisfying the ICCs (7).

3 Discrete-Time Systems

This section defines the ICC control problem for discrete-time systems and provides LMIs conditions to synthesize state-feedback and dynamic output-feedback controllers. Consider the following discrete-time system:

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p u(k) + D_p w_p(k), \\ y_p(k) &= C_p x_p(k), \\ z(k) &= M_p x_p(k) + v(k) \end{aligned} \tag{21}$$

Suppose that we apply to the plant (21) a full state-feedback stabilizing control

$$u(k) = Kx(k) \tag{22}$$

or a strictly proper stabilizing control

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c z(k) \\ u(k) &= C_c x_c(k) \end{aligned} \tag{23}$$

Then, the closed-loop system has the following form:

$$\begin{aligned} x(k+1) &= Ax(k) + Dw(k) \\ y(k) &= \begin{bmatrix} y_p(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} C_y \\ C_u \end{bmatrix} x(k) = Cx(k) \end{aligned} \tag{24}$$

where the definitions of matrices A, D, and C, and vectors $x(k)$, $w(k)$, and $y(k)$ are the same as in the continuous-time case.

As in the continuous-time case, let $W_p > 0$ and $W_v > 0$ denote symmetric matrices with dimensions equal to $w_p(k)$ and $z(k)$, respectively. Also, define $W = W_p$ if state-feedback (22) is used or W defined as in Eq. (5) if dynamic output-feedback (23) is used. Then, let $\bar{P}$ denote the closed-loop controllability Gramian from the input $W^{-1/2}w$. Since A is stable, $\bar{P}$ is given by

$$\bar{P} = A\bar{P}A' + DWD' \tag{25}$$

As in the continuous-time case, we seek a solution to the following optimal control problem.

Problem 2. *Discrete-time ICC control: find a state-feedback stabilizing controller (22) or a strictly proper output-feedback stabilizing controller (23) for the system (21) to minimize the ICC cost*

$$J_{\mathrm{ICC}} = \mathrm{trace}\left(QC_y\bar{P}C_y'\right), \quad Q > 0 \tag{26}$$

subject to the control covariance constraint

$$U_i = \Phi_i C_u \bar{P} C_u' \Phi_i' \le \bar{U}_i, \quad i = 1, 2, \ldots, m \tag{27}$$

where $\bar{P}$ is given by Eq. (25) and the matrices Φ_i for $i = 1, 2, \ldots, m$ are as in the continuous-time case appropriately selected projection matrices for each u_i corresponding to the constraint $\bar{U}_i$.

To formulate the LMIs for the discrete-time case, we use the extended $\mathcal{H}_2$ norm LMIs given by Theorem 1 of Refs. [19–23] as a starting point. The following theorem formulates the discrete-time ICC problem with a set of LMIs conditions.

Theorem 4. *Consider the closed-loop system (24). Given the input constraints $\bar{U}_i$ for $i = 1, 2, \ldots, m$, if there exists a matrix G and a symmetric matrix P such that the following LMIs are satisfied*:

$$\begin{bmatrix} P & AG & DW^{1/2} \\ G'A' & G + G' - P & 0 \\ W^{1/2}D' & 0 & \mathbf{I} \end{bmatrix} > 0 \tag{28}$$

$$p\begin{bmatrix} \bar{U}_i & \Phi_i C_u G \\ G'C_u'\Phi_i' & G + G' - P \end{bmatrix} > 0 \tag{29}$$

then the closed-loop system (24) is asymptotically stable with an input covariance bounded by

$$\bar{U}_i > \Phi_i C_u P C_u' \Phi_i' > \Phi_i C_u \bar{P} C_u' \Phi_i' = U_i, \quad \forall i = 1, 2, \ldots, m \tag{30}$$

and an ICC cost bounded by

$$\begin{aligned} \bar{J}_{\mathrm{ICC}} &= \mathrm{trace}\left(QC_y P C_y'\right) \\ &\ge \mathrm{trace}\left(QC_y\bar{P}C_y'\right) = J_{\mathrm{ICC}} \end{aligned} \tag{31}$$

Proof. The closed-loop system (24) is exponentially stable as a result of noticing that $DW^{1/2}$ is the weighted disturbance input matrix and by applying Theorem 1 of Ref. [19] to obtain the LMI (28). Since (28) implies that

$$P > APA' + DWD'$$

there exists a matrix $M = M' > 0$ such that

$$P = APA' + DWD' + M$$

Consequently, $P > \bar{P}$, where $\bar{P}$ satisfies Eq. (25), leads to

$$\mathrm{trace}\left(QC_yPC_y\right) > \mathrm{trace}\left(QC_y\bar{P}C_y\right)$$

such that Eq. (31) holds. The Schur complement of the LMI (29) can be written as

$$\bar{U}_i - \Phi_i C_u G(G+G'-P)^{-1}G'C_u'\Phi_i' > 0, \quad \forall i = 1,2,\ldots,m$$

In order for the LMI (29) to be feasible, the following inequality should be satisfied:

$$G + G' > P > 0$$

Thus, it follows that:

$$\begin{aligned}\bar{U}_i &> \Phi_i C_u G(G+G'-P)^{-1}G'C_u'\Phi_i' \\ &> \Phi_i C_u P C_u' \Phi_i' \\ &> \Phi_i C_u \bar{P} C_u' \Phi_i' = U_i\end{aligned}$$

for all $i = 1,2,\ldots,m$, such that (30) holds.

3.1 State Feedback Control. With the state-feedback controller (22), the closed-loop matrices are the same as in the continuous-time case (15).

THEOREM 5. *There exists a controller in the form (22), given by*

$$K = LG^{-1}$$

that minimizes J_{ICC} (26) and satisfies the input constraints (27), if there exist matrices $L \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times n}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ that minimize the upper bound of the ICC cost

$$\begin{aligned}\bar{J}_{\mathrm{ICC}} &= \min_{P,L,G} \mathrm{trace}\left(QC_pPC_p'\right) \\ &> \mathrm{trace}\left(QC_y\bar{P}C_y'\right) = J_{\mathrm{ICC}}\end{aligned}$$

subject to the LMIs

$$\begin{bmatrix} P & A_pG + B_pL & D_pW_p^{1/2} \\ \star & G+G'-P & 0 \\ \star & \star & \mathbf{I}\end{bmatrix} > 0 \tag{32}$$

$$\begin{bmatrix} \bar{U}_i & \Phi_i L \\ \star & G+G'-P\end{bmatrix} > 0 \tag{33}$$

for $i = 1,2,\ldots,m$.

Proof. Substituting the closed-loop matrices (15) into the LMI (28) of Theorem 4, results in the following bilinear matrix inequality:

$$\begin{bmatrix} P & (A_p + B_pK)G & DW^{1/2} \\ G'(A_p+B_pK)' & G+G'-P & 0 \\ W^{1/2}D' & 0 & \mathbf{I}\end{bmatrix} < 0$$

Multiplying the instrumental matrix variable G into the parenthesis gives

$$\begin{bmatrix} P & A_pG + B_pKG & DW^{1/2} \\ G'A_p' + G^TK^TB_p' & G+G'-P & 0 \\ W^{1/2}D' & 0 & \mathbf{I}\end{bmatrix} < 0$$

which clearly shows multiplication between the state-feedback controller K and the instrumental matrix variable G. However, by using the change of variables $L = KG$ [19], the LMI (32) is constructed. The same procedure is used to construct the LMI (33) as well. When the LMIs (32) and (33) are satisfied, the state-feedback controller K (asymptotically) stabilizes the closed-loop system while minimizing the upper bound cost $\bar{J}_{\mathrm{ICC}}$ and satisfying the ICCs (27).

3.2 Dynamic Output Feedback Control. Consider a strictly proper dynamic output-feedback controller defined in Eq. (23), the closed-loop matrices are the same as in the continuous-time (18).

THEOREM 6. *There exists a controller in the form (3), given by*

$$\begin{aligned} A_c &= V^{-1}\left(R - YA_pX - YB_pL - FM_pX\right)U^{-1} \\ B_c &= V^{-1}F \\ C_c &= LU^{-1}\end{aligned}$$

that minimizes J_{ICC} (26) and satisfies the input constraints (27), if there exist matrices $X \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{n\times n}, S \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{n\times n}, J \in \mathbb{R}^{n\times n}, L \in \mathbb{R}^{m\times n}$, and $F \in \mathbb{R}^{n\times q}$ and symmetric matrices $\mathscr{P} \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times n}$ that minimize the upper bound of the ICC cost

$$\begin{aligned}\bar{J}_{\mathrm{ICC}} &= \min_{\mathscr{P},J,H,X,Y,R,L,F,S} \mathrm{trace}\left(QC_pPC_p'\right) \\ &> \mathrm{trace}\left(QC_y\bar{P}C_y'\right) = J_{\mathrm{ICC}}\end{aligned}$$

subject to the LMIs

$$\begin{bmatrix} \mathscr{P} & J & A_pX + B_pL & A_p & B_w \\ \star & H & R & YA_p + FM_p & YB_w + FD_{zw} \\ \star & \star & X+X'-P & I+S'-J & 0 \\ \star & \star & \star & +Y'-H & 0 \\ \star & \star & \star & \star & \mathbf{I}\end{bmatrix} > 0 \tag{34}$$

$$\begin{bmatrix} \bar{U}_i & \Phi_i L & 0 \\ \star & X+X'-P & \mathbf{I}+S'-J \\ \star & \star & Y+Y'-H\end{bmatrix} > 0 \tag{35}$$

for $i = 1,2,\ldots,m$.

Proof. Substituting the closed-loop matrices (18) into the LMIs conditions of Theorem 4 results in nonlinear matrix inequalities. To obtain a set of LMI conditions, a nonlinear transformation (change-of-variables) is performed. First, the slack variable G, its inverse G^{-1}, the Lyapunov matrix P, and the controller matrix $\mathscr{K}$ are partitioned as

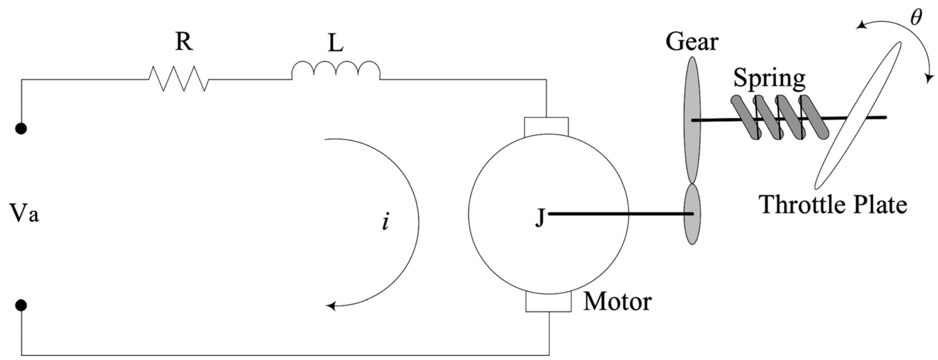

Fig. 1 An electronic throttle system

$$G := \begin{bmatrix} X & Z_1 \\ U & Z_2 \end{bmatrix}, \quad G^{-1} := \begin{bmatrix} Y' & Z_3 \\ V' & Z_4 \end{bmatrix},$$
$$P := \begin{bmatrix} \mathscr{P} & P_2 \\ P_2' & P_3 \end{bmatrix}, \quad \mathscr{K} := \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$$

Then, the transformation matrix

$$\mathcal{J} := \begin{bmatrix} I & Y' \\ 0 & V' \end{bmatrix}$$

used in Refs. [19,20,22] is introduced and the following nonlinear change of variables is defined:

$$\begin{bmatrix} R & F \\ L & 0 \end{bmatrix} := \begin{bmatrix} V & YB_u \\ 0 & I \end{bmatrix} \mathscr{K} \begin{bmatrix} U & 0 \\ C_y X & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A [X \quad 0],$$
$$\begin{bmatrix} \mathscr{P} & J \\ J' & H \end{bmatrix} := \mathcal{J}' P \mathcal{J},$$
$$S := YX + VU$$

After substituting the closed-loop matrices (18) into the LMIs conditions of theorem 4, then the following congruence transformations:

$$T_1 = \mathrm{diag}(\mathcal{J}, \mathcal{J}, I), \quad T_2 = \mathrm{diag}(I, \mathcal{J})$$

are applied to the LMIs (28) and (29) to obtain the LMIs (34) and (35), respectively. For example, the LMI (34) is obtained by multiplying the LMI (28) by T_1 on the right and T_1' on the left. Then as a result of applying Theorem 4, when the LMIs (34) and (35) are satisfied, the closed-loop system formed with the output-feedback controller (23) is asymptotically stable while minimizing the upper bound of the ICC cost $\bar{J}_{\mathrm{ICC}}$. Furthermore, the ICCs defined in Eq. (27) are satisfied.

4 Experimental Setup

To demonstrate the effectiveness of the LMIs-based controller synthesis presented in this paper, an ICC controller has been designed and validated experimentally on electronic throttle control (ETC) system. An electronic throttle establishes the essential connection between the acceleration pedal and the throttle valve using electronic signals instead of a mechanical link. The traditional engine throttle is mechanically connected to the vehicle acceleration pedal. The engine charge air quantity is controlled by the throttle plate position directly. The fuel quantity tracks the charge air to provide the desired air-to-fuel ratio, which is critical for engine emission regulation. The advantage of using the ETC is that the engine charge air and fuel can be regulated simultaneously, providing accurate air-to-fuel ratio control, especially under the transient engine operational conditions. A typical electronic throttle consists of a throttle body with an electric direct current (DC) motor, a pair of throttle position sensors (potentiometers), and a control unit. The output voltage of the potentiometer is proportional to the throttle position. Figure 1 illustrates the schematic diagram of the electronic throttle system. ETC parameters' definition is shown in Table 1.

It is well-known that the dynamic model of ETC system is nonlinear due to spring preload and friction torques [24]. Feedforward control term

$$u_0 = \frac{RT_{\mathrm{s}}}{K_{\mathrm{m}} V_{\mathrm{b}}} \mathrm{sgn}(\theta)$$

was used to compensate for this nonlinearity, with $\mathrm{sgn}(\theta)$ is the signum function of throttle angle. This nonlinear term will be removed from the system model (so that we can represent the system as a linear system), and it will be added back to the designed controller during simulation and experiment stage. Process noise $w_p(t)$ is weighted by $W_p = 1$. The motor voltage

$$V_{\mathrm{a}} = V_{\mathrm{b}} u(t)$$

is regulated by pulse width modulated (PWM) duty cycle of the control signal $u(t)$. Thus, the ETC system model can be represented by the following continuous-time state space model (see Ref. [25] for details on ETC modeling):

$$\dot{x}_p = A_p x_p(t) + B_p u(t) + D_p w_p(t),$$
$$y_p(t) = C_p x_p(t), \quad z(t) = M_p x_p(t)$$

where

$$A_p = \begin{bmatrix} 0 & 1 \\ -\dfrac{K_{\mathrm{s}}}{J} & -\dfrac{1}{J}\left(\dfrac{K_{\mathrm{a}} K_{\mathrm{m}}}{R} + K_{\mathrm{B}}\right) \end{bmatrix},$$
$$B_p = \begin{bmatrix} 0 \\ \dfrac{K_{\mathrm{a}} V_{\mathrm{b}}}{JR} \end{bmatrix}, \quad x_p(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix},$$
$$C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_p = C_p, \quad D_p = B_p$$

$y_p(t), z(t)$, and $w_p(t)$, and system matrices are defined as in Eq. (1). Due to severe coupling among throttle parameters and the associated nonlinearities, it is not reasonable to use experimental data for throttle step and sinusoidal responses to obtain all these coupled parameters. Therefore, suitably designed experiments have been conducted to isolate these parameters in order to facilitate their identification accurately. The spring preload torque and stiffness were obtained by measuring the torque at various (stationary) plate angles. The system inertia and viscous friction were

Table 1 Electronic throttle parameters' definition

Parameter	Definition
Θ	Throttle plate angle
J	Motor inertia
K_a	Motor torque coefficient
K_m	Back electric and magentic field coefficient
K_s	Spring stiffness
K_B	Viscous friction coefficient
V_a	Motor voltage
V_b	Battery voltage
R	Electrical resistance
L	Coil inductance
T_s	Spring preload torque

Table 2 Parameters of electronic throttle

Parameter	Value	Parameter	Value
R	2.07	J	0.0035
K_a	0.487	K_m	0.4889
K_s	0.0914	K_B	0.005

obtained together by releasing the throttle plate freely from different holding positions, recording correspondent responses, and conducting the simulation studies to find the best match for inertia and viscous friction coefficient. The identified throttle parameters are given in Table 2.

There are two approaches to design a discrete-time controller to be implemented into a microprocessor for a continuous-time plant. One is to design a continuous-time controller and then discretize it and the other is to discretize the plant and design a discrete-time controller directly. For the current paper, the second approach was used. Since the angular velocity (second state) is not available for measurement, reduced order estimator has been designed to estimate $x_2(t)$. Third virtual state has been augmented to the discrete-time model to include integral action such that zero tracking error can be achieved. Since the throttle position can be measured directly, this (additional) state can be obtained online by integrating the tracking error. Thus, all the states will be available by measurement, estimation, and integration to implement the full state-feedback ICC controller.

LMI synthesis conditions of Theorem 5 have been coded into MATLAB through the parser YALMIP [26] with the LMI solver SeDuMi [27] to obtain ICC controller with actuator constraint. The computation time required to obtain the optimal solution was a approximately 0.33 s using a computer with a 2.4 GHz Intel Core $i5$ central processing unit. The controller was designed with $\bar{U} = 7.5$ with a resulting J_{ICC} cost of 0.863.

This controller has been implemented experimentally into Opal-RT based prototype controller with a sampling time 0.001 s. The experimental setup is shown in Fig. 2 with block diagram illustrating connection between various modules. The system is composed of five basic parts: an electronic throttle body, an H-bridge drive circuit, an input/output communication box, an Opal-RT prototype controller, and a host computer. The controller provides both PWM magnitude and sign control signals to an H-bridge driver which is used to control the throttle DC motor.

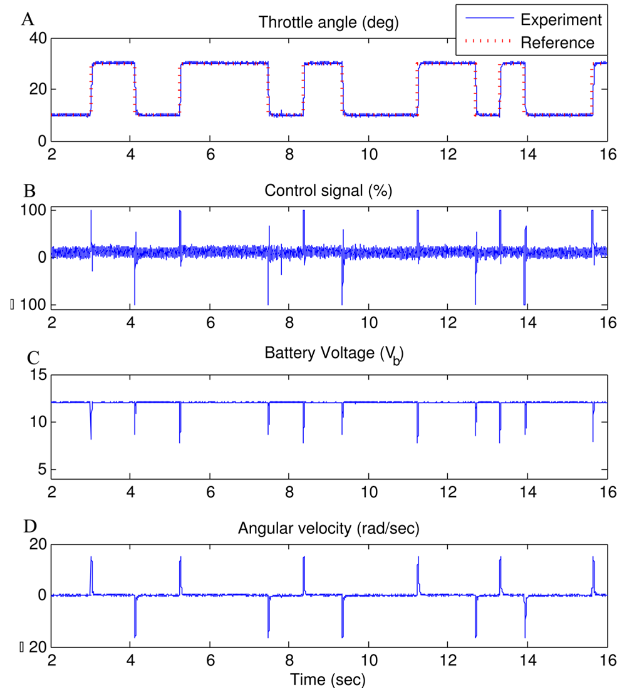

Fig. 3 Experimental tracking and signals of throttle the ICC control

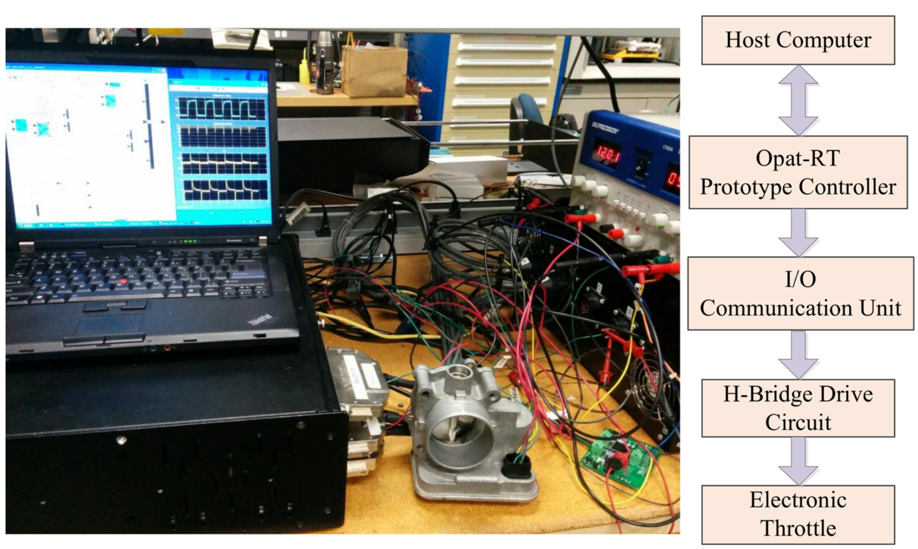

Fig. 2 Experiment test bench setup and block diagram

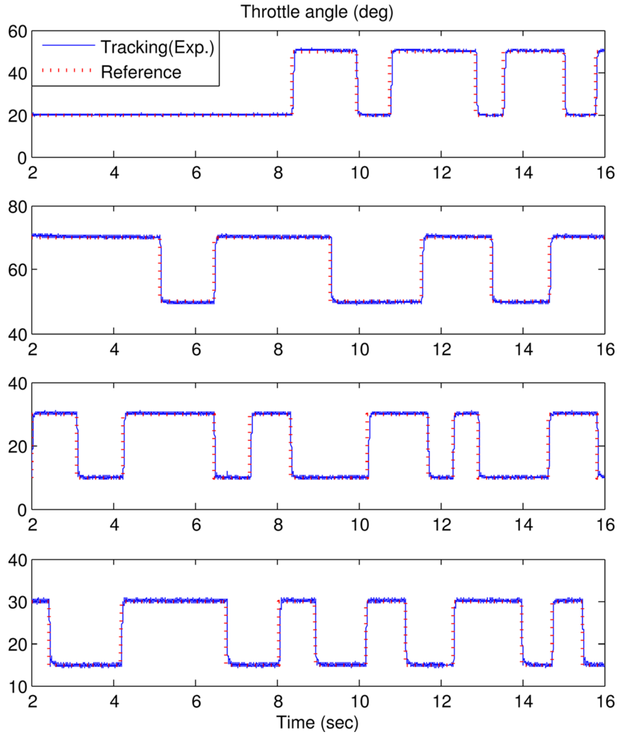

Fig. 4 Tracking experiment for different throttle angles

5 Experimental Results and Discussion

This section presents experimental results of the developed ICC controller for various throttle opening. Performance comparison with PID controller also included in this section as well.

Figure 3 illustrates throttle responses with respect to a series of step changes in the reference signal from 40 to 20 deg and vice versa. Figure 3(*a*) shows the good controller performance in terms of response speed and zero tracking error, while Fig. 3(*b*) illustrates the control signal. The battery voltage is shown in Fig. 3(*c*) which is used with the PWM signal $u(t)$ to regulate the DC motor voltage (V_a). The angular velocity of the throttle plate is shown in Fig. 3(*d*).

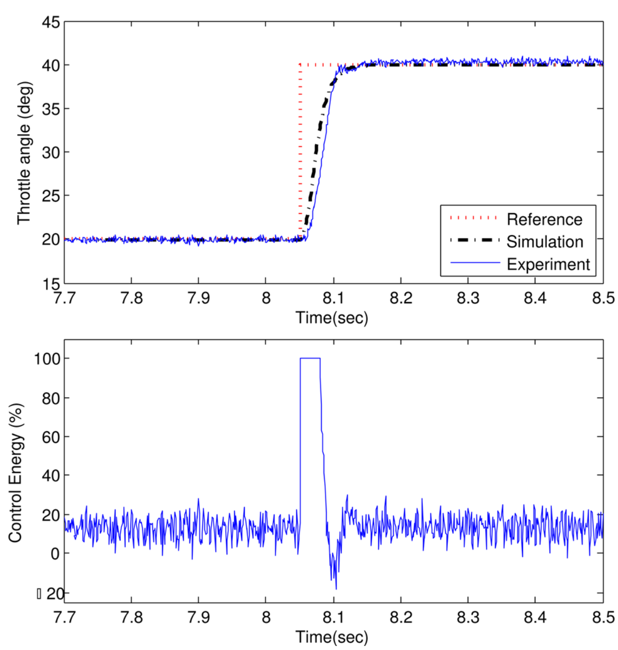

Fig. 5 Experiment and simulation tracking (raising edge)

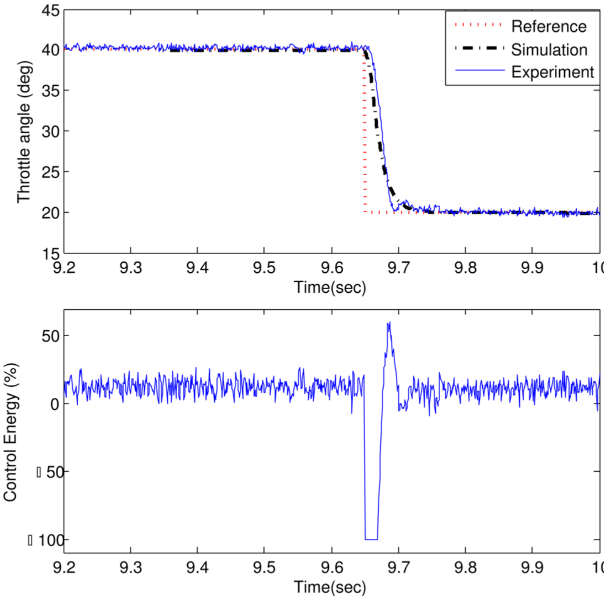

Fig. 6 Experiment and simulation tracking (falling edge)

Figure 4 validates experimental performance of the ICC controller for various throttle positions. It shows transitions from 20 deg to 50 deg, 70 deg to 50 deg, 30 deg to 10 deg, and 30 deg to 15 deg tracking performance, respectively.

Settling time is one of the most important time domain performance specifications in throttle control, since vehicle's acceleration is directly related to pedal and throttle position. Simulation and experimental results showed that the settling time for the ICC controller is 0.06 s while that for the PID controller is 0.125 s, which shows satisfactory performance as illustrated in Figs. 5 and 6 for raising and falling edges, respectively.

A well-tuned PID controller [28] has been implemented as a base-line controller to compare with the proposed ICC controller in Fig. 7. This figure compares the performance and control energy of the two controllers. The ICC controller shows not only better performance but also a lower control energy than that

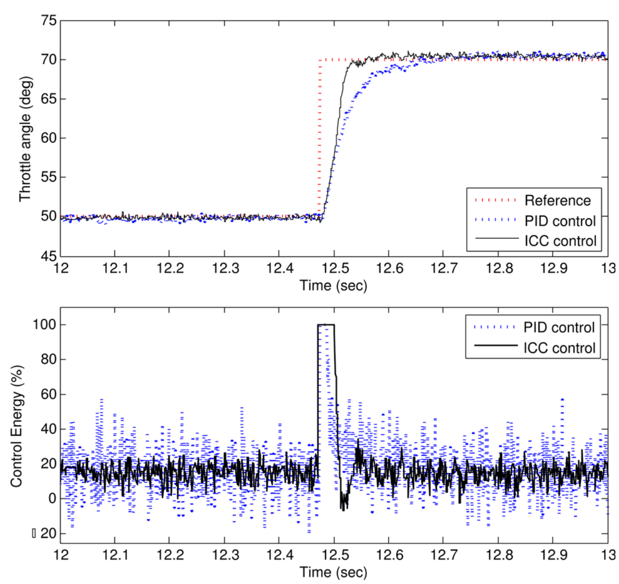

Fig. 7 Performance comparison between ICC and PID controllers

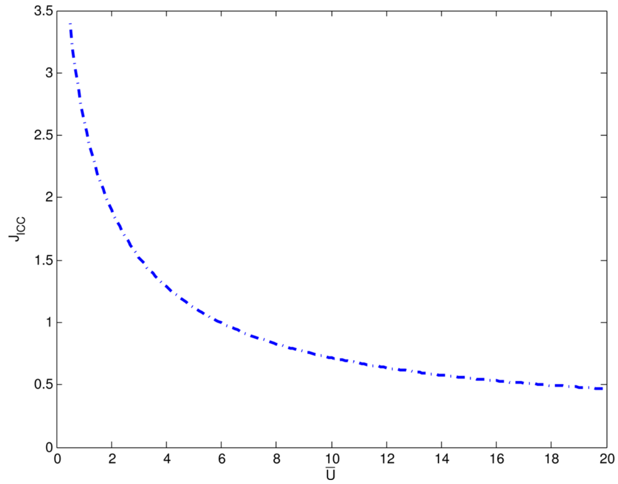

Fig. 8 Performance versus control energy

associated with the PID controller. The energy comparison of the control signals was performed in terms of $\mathcal{L}_2$ norm of the signals shown in the lower part of Fig. 7 as follows:

$$\begin{aligned} \| u_{\rm ICC} \|_2 &= \sqrt{\sum_{k=0}^{999} u_{\rm ICC}^2(k)} = 27.8428 \\ \| u_{\rm PID} \|_2 &= \sqrt{\sum_{k=0}^{999} u_{\rm PID}^2(k)} = 34.6172 \end{aligned} \tag{36}$$

This difference in control energy is based on the fact that the ICC controller accounts for the available control energy (in terms of $\bar{U}$) in the early design stage while the PID controller does not.

Finally, the relationship between control constraint ($\bar{U}$) and the achieved performance (J_{ICC}) is shown in Fig. 8. This figure has interesting interpretation, as actuator constraint become very tight (less than one), the output performance deteriorate considerably. On the other hand, as we relax the control constraint (increasing $\bar{U}$), a better output performance (lower ICC cost) will be obtained. In other words, the ICC controller takes into accounts the maximum available control energy in the early design stage to optimize the response subject to the available resources constraints.

6 Conclusion

In this paper, ICC control problem is solved via convex optimization with LMIs constraints for both continuous- and discrete-time systems. The theorems presented in this paper provide sufficient synthesis conditions based on LMI optimization scheme to obtain ICC controller, state-feedback, or dynamic output-feedback that solves the ICC problem. That is a controller that obtains the best possible output performance subject to multiple constraints on the input covariance matrices. As a practical engineering example, the control strategy was applied efficiently to an ETC system. An ICC controller for the ETC system was designed, implemented, and experimentally validated for throttle position control. The ICC controller was compared with a baseline PID controller and showed that the required control energy of the ICC controller is about 80% of the PID one in addition to the improved performance. Experimental results demonstrated that the LMIs-based controller synthesis conditions provided in this paper are capable of handing constraints on the input covariance (control effort) which are very common in practical engineering applications. It is worth to mention that while we have setup a single input constraint on the ETC system, the framework is capable of handling multiple constraints on different control channel simultaneously.

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