Proceedings of the 2006 American Control Conference
Minneapolis, Minnesota, USA, June 14-16, 2006

Synthesis of Multiple Robust Controllers for Parametric Uncertain LTI Systems

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Abstract— **This paper tackles the problem of designing multiple controllers that optimize the worst-case performance of a linear time invariant system under parametric uncertainty. The parametric uncertainty region is assumed to be convex polytopic, which is also partitioned into a set of convex polytopic local regions. It is desired that all plants that belong to a local region are to be controlled by a single controller, which is designed to give an optimal worst-case performance for that region. The *total performance* is evaluated as the maximum of worst-case performances for all the local regions. It is minimized with respect to a fixed number of convex polytopic local regions, as well as the same number of controllers. Even though the formulated problem is nonconvex, and thus it is difficult to ensure global optimality, algorithms are provided to update the local regions and the multiple controllers so that they guarantee monotonic non-increasing total performance.**

I. Introduction

Robust performance is of great importance when there is uncertainty in a plant to which performance is sensitive. It will give the information on worst-case performance that can happen due to some plant uncertainty. Many analysis and synthesis results have already been obtained for both robust $\mathcal{H}_2$ and robust $\mathcal{H}_\infty$ performance, mainly based on numerically efficient convex optimization (or LMI) techniques [1].

Most results on robust performance synthesis so far have been concerned with design of a *single* controller for all plants in the assumed model set. However, a single controller may not be sufficient to achieve the required performance, since the robustness requirement has to be met at the expense of performance limitations. Moreover, many standard controller design techniques, which find a common Lyapunov function and a single controller, may be infeasible if the size of uncertainty is large.

To overcome these drawbacks, in this paper, we will consider the design of *a set of controllers* that guarantee robust performance for parametric uncertain systems. Each controller in this set takes charge of a local region in the entire uncertainty region and is designed to give an optimal worst-cast performance for the specific region for which it is designed. The *total performance* of the control set is evaluated as the maximum of such worst-case performance in each local region. Therefore, the total performance of this set of multiple controllers is expected to be better than the worst-case performance of a single controller, designed for the entire region. In addition, segmenting an uncertainty region into smaller regions and utilizing multiple robust controllers is advantageous from the controller design perspective, since it is generally easier to obtain a global Lyapunov function for an uncertainty region, as the region size decreases.

Systems with multiple controllers have increasingly drawn the attention of control engineers in adaptive control or multiple modeling applications, in order to improve the controller performance under very different conditions [2], [3], [4]. However, most of the results so far have not paid enough attention to the importance on how to divide the entire uncertainty (or a set of different situations) into small partitioned regions. Here, we study how to simultaneously design a partition of the uncertainty and a set of robust controllers in order to improve the total performance of the control system in a systematic manner.

In this paper, minimization of the total performance is considered with respect to the fixed number of partitioned polytopic uncertainty subsets and controllers. Each partitioned uncertainty region is enforced to be convex polytopic and the controller of each partition is designed using a convex optimization technique. Unfortunately, the minimization problem turns out to be hard. Even for a given set of convex polytopic uncertainty regions, designing globally optimal full-order output-feedback controllers for polytopic uncertain systems is a nonconvex optimization problem [5]. Therefore, we present algorithms for updating both a partition of the uncertainty set and the set of multiple controllers that guarantee monotonic non-increasing total performance.

The resulting partition of an uncertainty set and robust controllers can be utilized in the following way. For each real plant, uncertain parameters are to be estimated either off-line or on-line. For off-line estimation, a local region that the estimated parameters lie will be detected, and a controller corresponding to that region will be selected and applied to the real plant. On the other hand, for on-line estimation, we need to use an initial robust controller designed for the entire uncertainty region during parameter identification [6], [7]. After finding the local region that contains an estimated parameter, a user can switch the controller to the particular controller that was specially designed for that region, to improve performance.

This paper is organized as follows. In Section II, we introduce several concepts and notations concerning uncertain systems that are necessary for the subsequent arguments. Using these concepts, Section III formulates a worst-case performance minimization problem with multiple controllers. For the formulated problem, in Section IV, algorithms are presented to design a partition and its multiple robust

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controllers simultaneously, which are locally optimal. The algorithms intertwine partitioning techniques of polytopic uncertain systems and a robust controller design method, both of which are also explained in Section IV. In Section V, we apply the algorithm to an example in order to illustrate the potential of multiple controllers with a properly chosen partition compared to a single controller. The initial robust $\mathcal{H}_\infty$ controller synthesis, which is important to our nonconvex optimization, is described in the Appendix.

II. Preliminaries

In this section, we define and review several concepts that will arise throughout the paper.

A. Notations

$\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. $\mathbb{R}^n$ is the set of real n dimensional vectors. $\mathbb{R}_+$ is the set of positive real numbers. $\mathbb{Z}_+$ is the set of positive integers. The interior of a set S is denoted as $\text{Int}(S)$. The Cartesian product of the indexed family of sets $\{S_i\}$ is denoted by $\prod_{i\in\mathbb{Z}_+} S_i$. The symbol $Sym(\cdot)$ denotes $Sym(A) = A + A^*$, where A^* is a complex conjugate transpose of A. $\mathbf{Co}\{A_1, \cdots, A_m\}$ represents the convex hull of a set of matrices $\{A_i\}$ and is defined as the intersection of all convex sets containing all elements of $\{A_i\}$. $\mathcal{V}(S)$ is the set of all the vertices of a convex polytope S. The direct sum of two matrices A and B is denoted as $A \oplus B := \text{diag}(A, B)$. The Kronecker product of two matrices $A \in \mathbb{R}^{n\times p}$ with entries a_{ij} and $B \in \mathbb{R}^{m\times k}$ is denoted by $A \otimes B$ and defined by the $mn \times kp$ block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1p}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{np}B \end{bmatrix}. \tag{1}$$

The transfer function of a continuous time linear time invariant (LTI) system with a realization is denoted by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B. \tag{2}$$

B. A partition of a parameter set

For a given compact and closed parameter set Λ, we call $\{\Lambda_q : q \in \mathcal{Z}_N\}$ *a partition* of a set Λ, where $\mathcal{Z}_N$ is the set of N coordinates of the partition, if $\Lambda_q \subset \Lambda$ is nonempty, closed and satisfies

$$\begin{gathered} \text{Int}(\Lambda_i) \cap \text{Int}(\Lambda_j) = \emptyset \,\text{for}\; i \neq j, \\ \bigcup_{q\in\mathcal{Z}_N} \Lambda_q = \Lambda. \end{gathered} \tag{3}$$

In words, a partition of a set Λ is a collection of nonempty subsets of Λ whose interior is disjoint and whose union is all of Λ.

C. Probability of a stochastic process

Define a random vector λ from a stochastic process $\lambda : \mathbb{Z}_+ \to \Lambda$ with a probability density function $f_\lambda : \Lambda \to \mathbb{R}_+$.

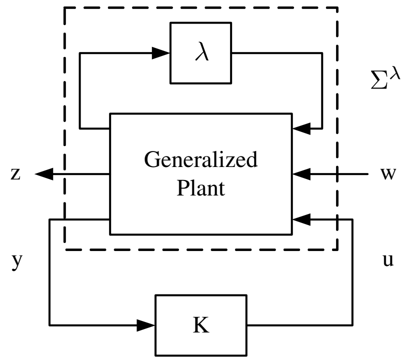

Fig. 1. A generalized plant with parametric uncertainty λ, (denoted by Σ^λ) and a controller K.

The probability of λ being contained in $\hat{\Lambda} \subset \Lambda$ is denoted as $Prob(\lambda \in \hat{\Lambda})$ and defined by the formula

$$Prob(\lambda \in \hat{\Lambda}) := \int_{\mu\in\hat{\Lambda}} f_\lambda(\mu)d\mu. \tag{4}$$

D. Polytopic parametric uncertain LTI systems

Let us consider a continuous time[1] linear time invariant generalized plant with parametric uncertainty $\lambda \in \Lambda$, in short denoted by Σ^λ, as depicted in Fig. 1,

$$\begin{bmatrix} z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A^\lambda & B_1^\lambda & B_2^\lambda \\ \hline C_1^\lambda & D_{11}^\lambda & D_{12}^\lambda \\ C_2 & D_{21} & 0 \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix},\; A^\lambda \in \mathbb{R}^{n\times n}, \tag{5}$$

where $z \in \mathbb{R}^{n_z}$ is the output of the system, $w \in \mathbb{R}^{n_w}$ is the disturbance to the system, $u \in \mathbb{R}^{n_u}$ is the control action and $y \in \mathbb{R}^{n_y}$ is the measured output. Here, the superscript "λ" on a matrix means that the matrix is a function of an uncertain parameter vector λ. The set of all uncertain plants with $\lambda \in \Lambda$ is denoted as Σ^Λ given by the formula

$$\Sigma^\Lambda := \left\{\Sigma^\lambda : \lambda \in \Lambda\right\}. \tag{6}$$

Assumptions on the plant parameters are as follow.

- A.1 $(A^\lambda, B_2^\lambda, C_2)$ is stabilizable and detectable for each $\lambda \in \Lambda$,
- A.2 Λ is a convex polytope.

The first assumption A.1 is necessary and sufficient for each system in Σ^Λ to be stabilizable with dynamic output feedback. A.2 is necessary for robust controller design based on LMI techniques.

E. Convexity of parametric uncertain LTI systems

Consider a set of LTI systems

$$\left\{\left[\begin{array}{c|c} A^\lambda & B^\lambda \\ \hline C^\lambda & D^\lambda \end{array}\right] : \lambda \in \Lambda\right\} \tag{7}$$

where $\Lambda \subset \mathbb{R}^u$. Correspondingly, we define a set of system matrices as

$$\left\{G^\lambda : \lambda \in \Lambda\right\} := \left\{\begin{bmatrix} A^\lambda & B^\lambda \\ C^\lambda & D^\lambda \end{bmatrix} : \lambda \in \Lambda\right\}. \tag{8}$$

[1]We only consider the continuous time case here but the application to the discrete time case is straightforward.

We assume that $\lambda \in \mathbb{R}^u$ and $G^\lambda \in \mathbb{R}^{n_1 \times n_2}$, where $n_1 := n + n_z + n_y$ and $n_2 := n + n_w + n_u$ is given by

$$G^\lambda = G_0 + (\lambda^T \otimes I_{n_1}) \begin{bmatrix} G_1 \\ \vdots \\ G_u \end{bmatrix}. \tag{9}$$

The following lemma is important for obtaining polytopic uncertain systems.

Lemma 1: If G^λ is affine in λ, and if Λ is a convex polytope, then the set of all the matrices $\{G^\lambda : \lambda \in \Lambda\}$ in Eq. (9) is a convex polytope in $\mathbb{R}^{n_1 \times n_2}$, and equivalent to the following set $\mathscr{G}^\Lambda$

$$\mathscr{G}^\Lambda := \mathbf{Co}\left\{G^\lambda : \lambda \in \mathcal{V}(\Lambda)\right\}. \tag{10}$$

The proof is straightforward and thus omitted here.

III. Problem formulation

Consider the set Σ^Λ of parametric uncertain LTI systems, each of which is represented in Eq. (5). We would like to design 1) a partition $\{\Lambda_q : q \in \mathcal{Z}_N\}$ of Λ, and 2) robust controllers $\{K_q : q \in \mathcal{Z}_N\}$, where K_q is applied to the set Σ^{Λ_q}, such that a certain worst-case performance cost (to be defined below) is minimized. The set of robust controller's coordinates is denoted as $\mathcal{Z}_N$ and let the coordinate of a robust controller be also the coordinate of the corresponding subset of the partition.

The regional controller K_q will be designed to regulate any linear uncertain system in the uncertain plant set Σ^{Λ_q}. Given a proper real-rational controller $K_q(s)$ for uncertain systems Σ^{Λ_q}

$$K_q(s) := \left[\begin{array}{c|c} A_{K_q} & B_{K_q} \\ \hline C_{K_q} & D_{K_q} \end{array}\right] \quad A_{K_q} \in \mathbb{R}^{n \times n}, \tag{11}$$

a realization of the closed loop transfer function from disturbance signal w to the output signal z is denoted by $T_{zw}(\lambda, K_q)$. We define several performance measures.

Definition 2: (Performance cost) For a given set of uncertain LTI systems Σ^{Λ_q} and a robustly stabilizing controller K_q, we define the *worst case performance cost* of the uncertain closed-loop systems by

$$J_i(\Lambda_q, K_q) := \sup_{\lambda \in \Lambda_q} \|T_{zw}(\lambda, K_q)\|_i, \tag{12}$$

where

$$\begin{cases} i = 2, & \mathcal{H}_2 \text{ norm performance,} \\ i = \infty, & \mathcal{H}_\infty \text{ norm performance.} \end{cases}$$

In the case when the probability density function f_λ is given, then the *averaged stochastic worst-case performance cost* can be defined by

$$EJ_i(\Lambda_q, K_q) := \int_{\lambda \in \Lambda_q} \sup_{\lambda \in \Lambda_q} \|T_{zw}(\lambda, K_q)\|_i f_\lambda(\lambda) d\lambda, \tag{13}$$

The *total performance* is defined as the maximum of such worst-case performance of each local region given by

$$\max_{q \in \mathcal{Z}_N} J_i(\Lambda_q, K_q), \tag{14}$$

for the worst-case performance case and by

$$\max_{q \in \mathcal{Z}_N} EJ_i(\Lambda_q, K_q), \tag{15}$$

for the averaged stochastic worst-case performance case.

Remark 3: From now on, we will deal mainly with the $\mathcal{H}_\infty$ performance (the induced $\mathcal{L}_2$ system gain) case, and the worst-case performance cost in Eq. (12). However, the results for the $\mathcal{H}_2$ performance case (or even mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance case [8]) and/or the averaged stochastic worst-case performance cost in Eq. (13) can be dealt with in an analogous manner.

We are interested in minimizing the total performance, which sets an upper bound on the performance of an individual closed-loop system. Next, the total performance minimization problem is formally stated.

Problem: For a given set of uncertain LTI systems Σ^Λ with a convex polytope Λ, design a partition $\{\Lambda_q : q \in \mathcal{Z}_N\}$ of the uncertainty set Λ and a set of stabilizing robust controllers $\{K_q : q \in \mathcal{Z}_N\}$, such that the worst-case $\mathcal{H}_\infty$ performance cost $J_\infty(\Lambda_q, K_q)$ in Eq. (12) for $\lambda \in \Lambda$ is minimized, i.e., solve the optimization problem

$$\min_{\{\Lambda_q\}, \{K_q \in \mathcal{K}(\Lambda_q)\}} \left[\max_{q \in \mathcal{Z}_N} J_\infty(\Lambda_q, K_q)\right], \tag{16}$$

where $\mathcal{K}(\Lambda_q)$ is the set of all controllers that exponentially stabilize the closed-loop system for all $\lambda \in \Lambda_q$.

The formulated optimization is nonconvex and therefore, it is difficult to solve exactly in the global optimal sense. In the following section, we will explain how to get a reasonable solution in a systematic way.

IV. Design of a partition and its multiple controllers

First we show how to partition a convex polytopic uncertainty set into a set of smaller convex polytopic uncertainty regions using the Cartesian product. The imposition of the constraint on a partition that it must have convex polytopic subsets, simplifies the tasks of determining partitions of uncertain systems as well as their respective sets of multiple robust controllers. Subsequently, we discuss the design of a set of robust controllers. Finally, a heuristic algorithm to find a local minimum for the optimization given by Eq. (16) will be presented.

The sampled uncertain system can be interpreted as follows. Recall $\lambda : \mathbb{Z}_+ \to \Lambda$ is an uncertain parameter of a linear system. For example, we may think of $i \in \mathbb{Z}_+$ as a serial number of the product (or plant) in the manufacturing line. We assume that the random vector λ is available and sampled at the end of production line or by a recursive online parameter estimator. The resulting partition $\{\Lambda_q : q \in \mathcal{Z}_N\}$ and its multiple controllers $\{K_q : q \in \mathcal{Z}_N\}$ are utilized by the selection function $\sigma(\cdot) : \Lambda \to \mathcal{Z}_N$ defined by the formula

$$\sigma(\lambda) = q, \quad \text{if } \lambda \in \Lambda_q \text{ for } q \in \mathcal{Z}_N. \tag{17}$$

After finding the local region that contains an estimated parameter, a user can switch using the selection rule $\sigma(\cdot)$

in Eq. (17) from a robust controller for the entire uncertain region to a controller for the local region to improve performance.

A. Partitioning polytopic uncertainty

Recall the uncertain parameter $\lambda(\cdot) : \mathbb{Z}_+ \to \Lambda$. Suppose that Λ consists of two parameter subspaces, one of which is to be partitioned while the other is not, i.e., $\lambda := [\lambda^{*T} \lambda_*^T]^T \in \Lambda \subset \mathbb{R}^u$, where $\lambda^* \in \Lambda^* \subset \mathbb{R}^{\bar{u}}$, $\lambda_* \in \Lambda_* \subset \mathbb{R}^{\underline{u}}$ and $\bar{u} + \underline{u} = u$. The partitioning of Λ^* is assumed to be of importance to enhance the performance of the closed-loop system, i.e., there is significant improvement if we partition Λ^* into multiple subsets. On the other hand, the parameter $\lambda_* \in \Lambda_*$ is not measured or estimated, since it may not be possible to estimate this parameter and/or it may not significantly affect the robust performance of the closed-loop system.

Now we present a useful technique for partitioning the entire uncertainty into a collection of convex polytopes: the Cartesian product partition technique.

Cartesian product partition

Suppose the parameter set is contained in a hyper-rectangular

$$\Lambda^* \subseteq \prod_{i=1}^{\bar{u}} [\lambda_i^{min}, \lambda_i^{max}], \tag{18}$$

where $\Lambda^* \in \mathbb{R}^{\bar{u}}$. Let each closed interval be partitioned as

$$\lambda_i^{min} = \delta_0^i < \delta_1^i < \cdots < \delta_{N_i - 1}^i < \delta_{N_i}^i = \lambda_i^{max}, \tag{19}$$

where N_i is the number of sections in ith coordinate of Λ^*. Define the coordinate of the subsets in the partition (and its respective controllers) as a $\bar{u}$-tuple of positive integers

$$q := (q_1, \cdots, q_{\bar{u}}) \in \mathcal{Z}_N \subset \mathbb{Z}_+^{\bar{u}}, \tag{20}$$

where $\mathcal{Z}_N$ is denoted by

$$\mathcal{Z}_N := \prod_{i=1}^{\bar{u}} \mathcal{Z}_{N_i}, \tag{21}$$

to be the set of all $\bar{u}$-tuples of elements of q such that $q_i \in \mathcal{Z}_{N_i} := \{1, 2, \cdots, N_i\}$ for each i. The number of total subsets in the parametric uncertainty partition is $N := N_1 \cdot N_2 \cdots N_{\bar{u}}$. Finally, we define the intersection of the cartesian product of the indexed family and Λ^*, denoted by Λ_q^*

$$\Lambda_q^* := \prod_{i=1}^{\bar{u}} [\delta_{q_i - 1}^i, \delta_{q_i}^i] \cap \Lambda^* \tag{22}$$

to be the set of all $\bar{u}$-tuples of elements of Λ^* such that $\lambda_i \in [\delta_{q_i-1}^i, \delta_{q_i}^i]$ and $\lambda_i \in \lambda_i^*$, where λ_i^* is the ith element of λ^*, for each i. Notice that $\Lambda_q := \Lambda_q^* \times \Lambda_*$ is a nonempty closed convex set. Λ_q's are illustrated in Fig. 2. By choosing appropriate δ_j^i in Eq. (19), Λ_q satisfies the conditions in Eq. (3): $\mathrm{Int}(\Lambda_i) \cap \mathrm{Int}(\Lambda_j) = \emptyset$ for $i \neq j$ and $\bigcup_{k=1}^N \Lambda_k = \Lambda$ for $q \in \mathcal{Z}_N$. The elements in a subset of borders $\{\delta_k^i, : k \in \mathcal{Z}_{N_i}\} \setminus \{\delta_0^i, \delta_{N_i}^i\}$ for $i \in \{1, \cdots, \bar{u}\}$ in Eq. (19) are the optimization parameters.

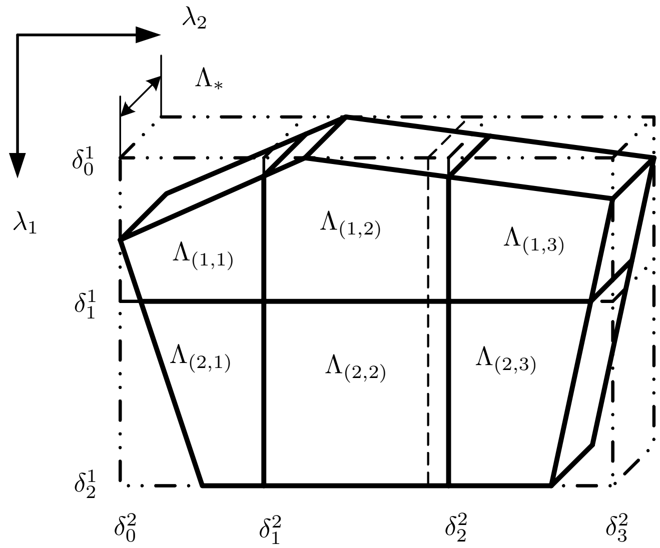

Fig. 2. A Cartesian product partition (solid line) and a locally updated partition (dotted line) for optimization.

B. Robust $\mathcal{H}_\infty$ controller design for a fixed partition

For a given convex polytopic subset Λ_q, a sub-optimal robust controller shall be designed. Let us parameterize the qth controller K_q in Eq. (11) by defining a matrix Θ_q as

$$\Theta_q := \begin{bmatrix} A_{K_q} & B_{K_q} \\ C_{K_q} & D_{K_q} \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times(n+n_y)} \tag{23}$$

Then, as in [9] the closed-loop system matrix with $\lambda \in \Lambda_q$ can be written in terms of Θ_q

$$\begin{bmatrix} A_{cl}^\lambda & B_{cl}^\lambda \\ C_{cl}^\lambda & D_{cl}^\lambda \end{bmatrix} = \begin{bmatrix} A_0^\lambda & B_0^\lambda \\ C_0^\lambda & D_{11}^\lambda \end{bmatrix} + \begin{bmatrix} \mathscr{B}^\lambda \\ \mathscr{D}_{12}^\lambda \end{bmatrix} \Theta_q \begin{bmatrix} \mathscr{C} & \mathscr{D}_{21} \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned} A_0^\lambda &:= \begin{bmatrix} A^\lambda & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}; \quad B_0^\lambda := \begin{bmatrix} B_1^\lambda \\ 0_{n\times n_w} \end{bmatrix}; \\ C_0^\lambda &:= \begin{bmatrix} C_1^\lambda & 0_{n_z\times n} \end{bmatrix}; \quad \mathscr{B}^\lambda := \begin{bmatrix} 0_{n\times n} & B_2^\lambda \\ I_{n\times n} & 0_{n\times n_u} \end{bmatrix}; \\ \mathscr{C} &:= \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ C_2 & 0_{n_y\times n} \end{bmatrix}; \quad \mathscr{D}_{12}^\lambda := \begin{bmatrix} 0_{n_z\times n} & D_{12}^\lambda \end{bmatrix}; \\ \mathscr{D}_{21} &:= \begin{bmatrix} 0_{n\times n_w} \\ D_{21} \end{bmatrix}. \end{aligned} \tag{25}$$

Now we present an algorithm for finding a robustly stabilizing controller matrix Θ_q that solves

$$\min_{\Theta_q} \max_{\lambda\in\Lambda_q} \|T_{zw}(\lambda, \Theta_q)\|_\infty. \tag{26}$$

This can be solved by optimization over a finite number of matrix inequalities [5]:

$$\begin{aligned} &\min_{P_q, \Theta_q, \gamma_q} \gamma_q, \quad \text{subject to} \\ &\mathscr{M}(P_q, \Theta_q, \lambda, \gamma_q) \succ 0 \end{aligned} \tag{27}$$

for $\lambda \in \mathcal{V}(\Lambda_q)$, where

$$\mathscr{M}(P_q, \Theta_q, \lambda, \gamma_q) := \begin{bmatrix} -Sym[P_q(A_0^\lambda + \mathscr{B}^\lambda \Theta_q \mathscr{C})] & P_q(B_0^\lambda + \mathscr{B}^\lambda \Theta_q \mathscr{D}_{21}) & (C_0^\lambda + \mathscr{D}_{12}^\lambda \Theta_q \mathscr{C})^T \\ \star & \gamma_q I & -(D_{11}^\lambda + \mathscr{D}_{12}^\lambda \Theta_q \mathscr{D}_{21})^T \\ \star & \star & \gamma_q I \end{bmatrix} \oplus P_q,$$

where $\star$ represents entries which follow from symmetry and P_q is a matrix of appropriate size satisfying $P_q = P_q^T$. Due to the uncertain parameter λ in the closed-loop system matrix in Eq. (24) and Lemma 1, the closed-loop matrix as in Eq. (24) is convex for $\lambda \in \Lambda_q$, therefore, the infinite number of LMIs $\mathscr{M}(P_q, \Theta_q, \lambda, \gamma_q) \succ 0$, where $\lambda \in \Lambda_q$ become a finite number of LMIs $\mathscr{M}(P_q, \Theta_q, \lambda, \gamma_q) \succ 0$, where $\lambda \in \mathcal{V}(\Lambda_q)$. This problem is however nonconvex, due to the coupling term between P_q and Θ_q in Eq. (27). A similar descent algorithm to the one used in [10], which utilizes the coordinate descent method for designing a $\mathcal{H}_2$ controller, is adopted for finding a local minimum for obtaining our $\mathcal{H}_\infty$ controllers. The procedure is as follows.

1) Initial design of Θ_q: This is explained in Appendix VI-A. Set the result of the initial design to Θ^0. Also set $j = 1$.
2) Design of P_q: Fix $\Theta_q := \Theta_q^j$. Solve the convex optimization problem in Eq. (27) with respect to γ_q and P_q. Set a solution P_q to P_q^j.
3) Design of Θ_q: Fix $P_q := P_q^j$. Solve the convex optimization problem in Eq. (27) with respect to γ_q and Θ_q. Set a solution Θ_q to Θ_q^{j+1}. Increment j by one. Continue this iteration until γ_q converges.

Since the value γ_q has a lower bound which is 0 and is monotonically non-increasing during the iterations, it will converge to some positive number. A robust $\mathcal{H}_\infty$ controller covering the entire uncertainty Λ can be obtained in the same way with Eq. $\mathscr{M}(P, \Theta, \lambda, \gamma) \succ 0$, where $\lambda \in \mathcal{V}(\Lambda)$.

C. Design of a partition with its respective set of robust controllers

Our main problem is to solve the following optimization:

$$\begin{aligned} &(\{\Lambda_q^{opt}\}, \{\Theta_q^{opt}\}) \\ &:= \arg \min_{\{\Lambda_q\}, \{\Theta_q \in \mathcal{K}(\Lambda_q)\}} \left[\max_{q \in \mathcal{Z}_N} J_\infty(\Lambda_q, \Theta_q) \right]. \end{aligned} \quad (28)$$

The following assumptions are introduced to develop an algorithm for solving the optimization problem in Eq. (28).

B.1 A partition of the uncertainty set $\{\Lambda_q\}$ is well-chosen so that the optimization in Eq. (27) is feasible for a given subset Λ_q. Moreover, the nonconvex optimization problem in Eq. (27) can be solved, and we denote $\hat{\Theta}_q$ as the *optimal* solution for a given subset Λ_q.

B.2 We denote $\hat{\gamma}_q$ as the performance cost achieved in Eq. (27) by the *optimal* controller $\hat{\Theta}_q$:

$$\hat{\gamma}_q := \max_{\lambda \in \Lambda_q} \|T_{zw}(\lambda, \hat{\Theta}_q)\|_\infty. \quad (29)$$

Moreover, we assume that $\hat{\gamma}_q$ continuously changes with respect to the changes in the subset Λ_q.

We now denote Λ_q^j as a given qth region of the partition at the iteration time j, and $\hat{\Theta}_q^j$ and $\hat{\gamma}_q^j$ (or $\hat{\gamma}(\Lambda_q^j)$) as the corresponding optimal controller and resulting performance cost respectively. Then we have a following fact.

Lemma 4: Let $\hat{\gamma} : \Lambda \to \mathbb{R}_+$ be a function from a given partitioned subset in Λ to the optimized performance cost in $\mathbb{R}_+$, then the function $\hat{\gamma}$ is continuous and monotone, i.e., whenever $\Lambda_q^j \subseteq \Lambda_q^{j+1}$ then $\hat{\gamma}(\Lambda_q^j) \le \hat{\gamma}(\Lambda_q^{j+1})$.

Proof: Continuity is given by assumption B.2. Monotonicity is guaranteed by the nature of the optimization, due to the inclusion relationship $\Lambda_q^j \subseteq \Lambda_q^{j+1}$. □

Now we present an algorithm to solve the main optimization problem in Eq. (28).

1) Design an initial partition and its respective set of controllers. Set the initial designs to $\{\Lambda_q^0\}$ and $\{\Theta_q^0\}$. Also set the performance cost in Eq. (27) to $\{\gamma_q^0\}$ and set $j = 0$.
2) Updating $\{\Lambda_q\}$ and $\{\Theta_q\}$: Update the partition $\{\Lambda_q^j\}$ so that the worst performance region should be reduced. Denote the set of coordinates of partitioned regions that produce the maximum total performance cost by

$$L^j := \left\{ \arg \max_{q \in \mathcal{Z}_N} \gamma_q^j \right\}. \quad (30)$$

Take out as many elements as possible from these subsets $\{\Lambda_q^j : q \in L^j\}$ and assign them to the neighboring subsets by adjusting borders, subject to the constraint that any updated partitioned subset must be either

$$\Lambda_q^{new} \supseteq \Lambda_q^j, \quad \text{or} \quad \Lambda_q^{new} \subseteq \Lambda_q^j, \quad q \in \mathcal{Z}_N. \quad (31)$$

Do this until $L^{new} \neq L^j$ after redesigning regional robust controllers. Set the new partition and its controllers to $\{\Lambda_q^{j+1}\}$ and $\{\Theta_q^{j+1}\}$.
3) Repeat this procedure: Compute the set L^j and update the partition and its controllers. In this way we can improve the total performance cost at each iteration. Continue this iteration until $\gamma_{q \in L^j}^j$ converges.

Remark 5: If the worst-case performance in the optimization problem in Eq. (28) is replaced by the averaged stochastic worst-case performance cost,

$$\begin{aligned}
&\min_{\{\Lambda_q\},\{\Theta_q\in\mathcal{K}(\Lambda_q)\}}\left[\max_{q\in\mathcal{Z}_N} EJ_\infty(\Lambda_q,\Theta_q)\right]\\
&=\min_{\{\Lambda_q\},\{\Theta_q\}}\left[\max_{q\in\mathcal{Z}_N}\int_{\lambda\in\Lambda_q}\sup_{\lambda\in\Lambda_q}\|T_{zw}(\Sigma^\lambda,\Theta_q)\|_\infty f_\lambda(\lambda)d\lambda\right]\\
&\le\min_{\{\Lambda_q\},\{\Theta_q\}}\left[\max_{q\in\mathcal{Z}_N}\int_{\lambda\in\Lambda_q}\hat{\gamma}_q f_\lambda(\lambda)d\lambda\right]\\
&=\min_{\{\Lambda_q\},\{\Theta_q\}}\left[\max_{q\in\mathcal{Z}_N}\underbrace{\hat{\gamma}_q Prob(\lambda\in\Lambda_q)}_{:=J^*(\Lambda_q,\Theta_q)}\right]
\end{aligned} \tag{32}$$

the upper bound of this performance cost can be minimized via a similar algorithm when the probability density function f_λ is continuous. $J^*(\Lambda_q,\Theta_q)$ defined in Eq. (32) plays the same role of $J_\infty(\Lambda_q,\Theta_q)$ in Eq. (28).

Unfortunately, these optimization problems are nonconvex and assumptions B.1 and B.2 may not be satisfied. First, given the current state of robust control theory, we can not compute the optimal solution $\hat{\Theta}_q$ for a given polytopic uncertain set Λ_q [5]. Therefore, in this paper, only a sub-optimal controller is computed as explained in the previous section. Therefore, the continuity assumption B.2 may be violated as well.

Given the Cartesian product partition technique previously described, it is easy to update the partition subject to the inclusion and exclusion constraint Eq. (31) as shown in Fig. 2. For example, assume that $q=(2,1)$ has the maximum performance cost among $q\in\mathcal{Z}_N$ then in the next iteration, $\gamma_{q=(2,1)}$ and $\gamma_{q=(2,2)}$ are expect to decrease while $\gamma_{q=(3,1)}$ and $\gamma_{q=(3,2)}$ are expected to increase.

The presented optimization problem in Eq. (16) is somewhat similar to the encoder and decoder design problem in the vector quantization theory [11].

V. An Illustrative Example

Consider a second order mass-spring-damper system that often arises in micro-machined inertial sensor systems, as an illustrative example. The state space model is given by

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Dw(t),\end{aligned} \tag{33}$$

where

$$\begin{aligned}
&A:=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix};\quad B:=\begin{bmatrix}0\\1\end{bmatrix};\\
&C:=\begin{bmatrix}1 & 0\end{bmatrix};\quad D:=1;\\
&\frac{k}{m}\in\left\{4\times10^5(1+w_k\lambda_1):w_k=0.5,|\lambda_1|\le1\right\}\\
&\frac{b}{m}\in\left\{4(1+w_b\lambda_2):w_b=0.5,|\lambda_2|\le1\right\}.
\end{aligned} \tag{34}$$

w_k and w_b represent 50% parameter variations. We select a performance weighting function $W_p(s)=g/(s+w_c)$, where $w_c=90\times2\pi$ and $g=1.1310\times10^4$ with the state space model $\dot{z}(t)=-w_cz(t)+gy(t)$ to achieve a disturbance rejection $1:20$ for constant disturbances from w to y and a bandwidth of around 10^4 (rad/s) for all possible values of the uncertain parameter. This performance requirement can be also viewed as a good tracking response of the closed-loop system from the input signal (or the force on mass) to the output signal (or the position of mass). Combine Eq. (33) and $W_p(s)$ into a generalized plant as

$$\begin{aligned}
\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix}&=\begin{bmatrix}A & 0_{2\times1}\\ gC & -w_c\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}0_{2\times1}\\ gD\end{bmatrix}w(t)\\
&\quad+\begin{bmatrix}B\\0\end{bmatrix}u(t)\\
z(t)&=\begin{bmatrix}0_{1\times2} & 1\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}\\
y(t)&=\begin{bmatrix}C & 0_{2\times1}\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}w(t).
\end{aligned} \tag{35}$$

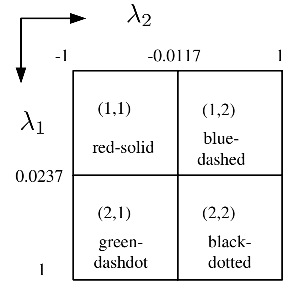

Fig. 3. The final partition, its coordinates and a legend for the randomly generated parameters.

TABLE I

Worst-case costs achieved by the multiple robust controllers.

$\gamma_{(1,1)}=0.9109$	$\gamma_{(1,2)}=0.9291$
$\gamma_{(2,1)}=0.9390$	$\gamma_{(2,2)}=0.9151$

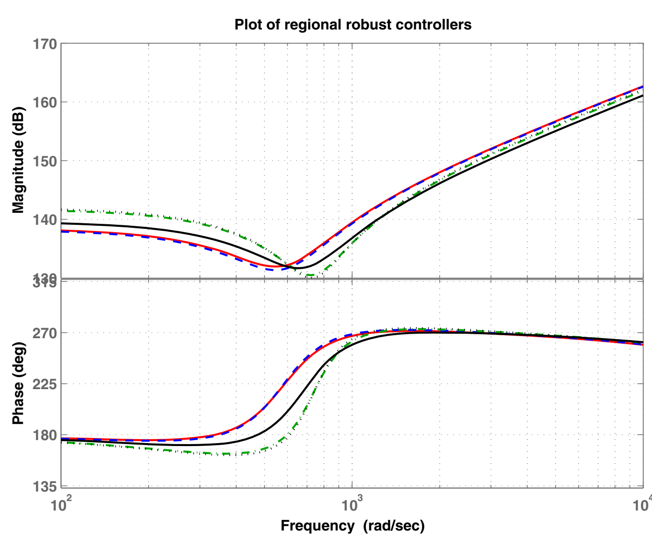

Fig. 4. *The single controller* K_{single} (black-solid line) and *the multiple controllers* $\{K_q:q\in\mathcal{Z}_4\}$ (see the legend in Fig. 3).

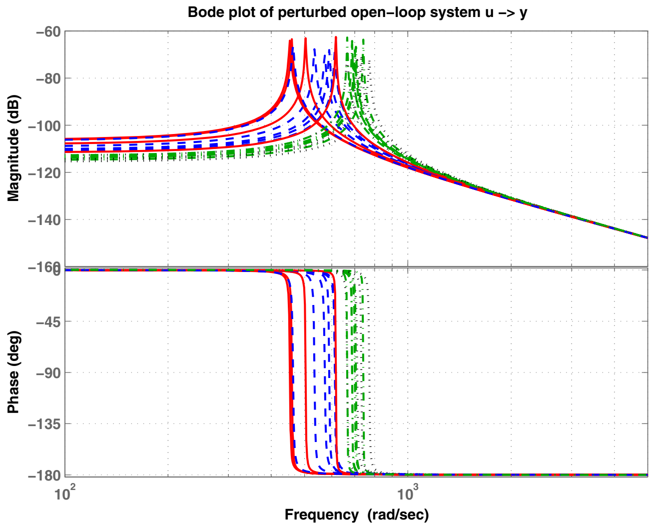

Fig. 5. Bode plot of the perturbed open-loop systems from u to y (see the legend in Fig. 3).

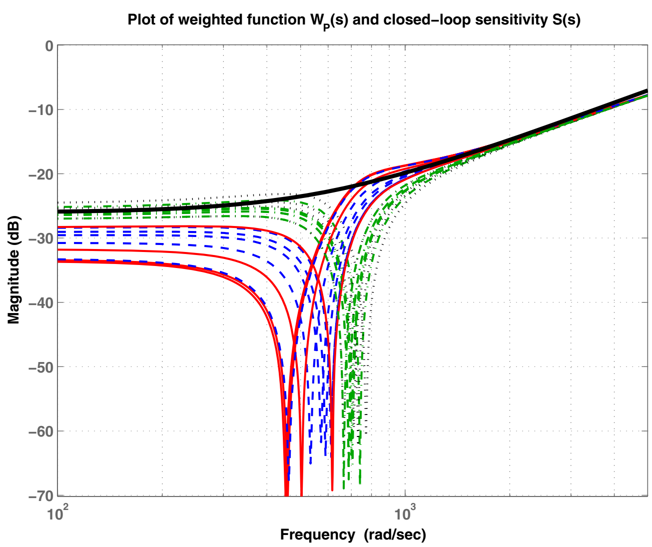

Fig. 6. Sensitivity functions of the perturbed closed-loop systems (see the legend in Fig. 3) with *the single controller* and the inverse of the weighting function W_p (black-solid).

For this example, we found a *single* robust controller K_{single} achieving an upper bound of $\gamma_{single} = 1.1845$ for the entire uncertainty region using the algorithm explained in the section IV-B. Next we found a Cartesian product partition (two by two) as shown in Fig. 3 and its *multiple* controllers $\{K_q : q \in \mathcal{Z}_N\}$ using the optimization algorithm elaborated in the section IV-C. The previously found single controller K_{single} served as initial controllers and the equally spaced partition was chosen as the initial partition for the nonconvex optimization as in Eq. (28). A subset of the partition can be arbitrarily reduced by the presented nonconvex optimization algorithm, which highly depends on initial conditions and the problem structure. The upper bounds achieved by multiple controllers are shown in Table I with respect to the coordinates of the partition. The iterations

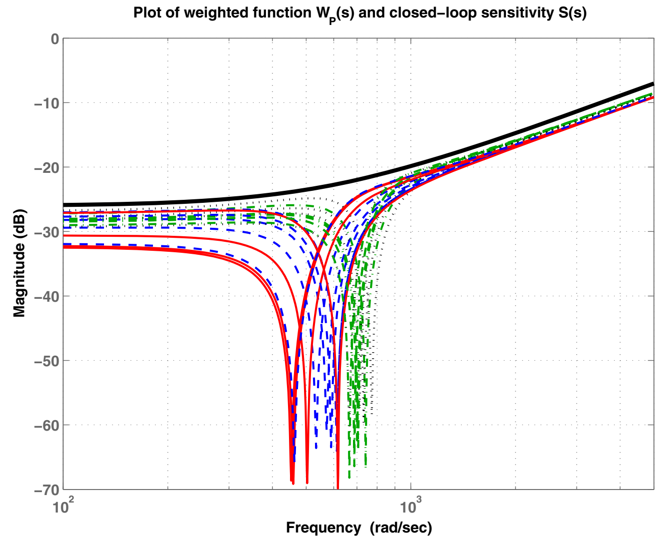

Fig. 7. Sensitivity functions of the perturbed closed-loop systems (see the legend in Fig. 3) with *the multiple controllers* and the inverse of the weighting function W_p (black-solid).

stopped when there were two big diagonal cost values such as $\gamma_{(1,2)} = 0.9291$ and $\gamma_{(2,1)} = 0.9390$, which blocks future improvement of the total cost by changing borders in any direction within specified tolerance. As shown in Table I, the total performance cost is $\gamma_{total} := \max_{q \in \mathcal{Z}_4} \gamma_q = 0.9390$, which improves the cost obtained by the single controller $\gamma_{single} = 1.1845$ by 20.73%. The bode-diagram of the single controller and those of the four multiple controllers are plotted in Fig. 4. It is very interesting to notice that $K_{(1,1)} \approx K_{(1,2)}$ and $K_{(2,1)} \approx K_{(2,2)}$, which hints of the enhanced significance of partitioning the λ_1 space (or spring constant parameter space) as compared to the λ_2 space (or damping coefficient parameter space). The single controller K_{single} is somewhat between these two groups. Starting from the single controller as the initial controllers, multiple controllers are transformed and optimized through iterations according to the partitioned uncertain subsets.

In order to evaluate the closed-loop uncertain systems, twenty perturbed systems were generated by parameters sampled from a stochastic process with a probability density function $f_\lambda(\nu_1, \nu_2) = f_{\lambda_1}(\nu_1) f_{\lambda_2}(\nu_2)$, where $f_{\lambda_1}(\nu) = f_{\lambda_2}(\nu) = 1/2$, if $\nu \in [-1, 1]$ otherwise $f_{\lambda_1}(\nu) = f_{\lambda_2}(\nu) = 0$, (i.e., it consists of two independent uniform probability density functions). Each randomly generated open-loop plant from the input signal u to the output signal y is classified and plotted according to the subset of the partition where it is contained as shown in Fig. 5.

- The four plants that belong to $\Lambda_{(1,1)}$ are plotted with red solid lines,
- the six plants that belongs to $\Lambda_{(1,2)}$ are plotted with blue dashed lines,
- the six plants that belong to $\Lambda_{(2,1)}$ are plotted with green dash-dotted lines,
- and the four plants that belong to $\Lambda_{(2,2)}$ are potted with black dotted lines, as summarized in Fig. 3.

These closed-loop sensitivity functions $S(s)$ from the disturbance d to the controlled output y by the single controller K_{single} are shown in Fig. 6 as compared to the corresponding multiple controllers $\{K_q : q \in \mathcal{Z}_4\}$ which are shown in Fig. 7. Some of the sampled $S(s)$ with the single controller do not meet the design specification, as depicted in Fig. 6. The design specification is satisfied with the partition and its multiple controllers and the sampled closed-loop sensitivity transfer functions all lie below the Bode magnitude plot of $W_p^{-1}(s)$, i.e., $\|W_pS\|_\infty < 1$ as shown in Fig. 7.

VI. Conclusions

In this paper, we presented an algorithm to determine a convex polytopic partitioning of an uncertainty region, and the design of a set of robust controllers for LTI systems with parametric uncertainty $\lambda \in \Lambda$, where Λ is convex polytope. Each controller takes charge of a local region and is designed to give a sub-optimal worst-cast performance for that region. Minimization of the maximum of such worst-case performance of each local region is considered with respect to a partition and its multiple robust controllers. The presented algorithm updates both partitions and controllers to achieve monotonic non-increasing total performance. An illustrative example was studied to evaluate the algorithm and to demonstrate the benefits of partitioning the uncertainty space and utilizing multiple robust controllers. As for future work, online/offline parameter estimation algorithms for the correct selection shall be explored as well as improved partitioning algorithms. Extensions from LTI parametric uncertain systems to linear parameter-varying (LPV) systems are quite natural and currently under investigation.

Appendix

A. Initial controller design in robust $\mathcal{H}_\infty$ synthesis

For nonconvex optimization, initial starting point is critical. We review a reasonable method for selection of an initial point proposed in [5].

1) State feedback: First, for the uncertain system Eq. (5), we design a state feedback controller: $u = Fx$ that optimizes $\mathcal{H}_\infty$ norm of the closed loop system: $\sup_{\lambda\in\Lambda_q} ||T_{zw}^\lambda||_\infty$. As in Theorem 6 in [5], this can be solved by convex optimization with LMIs.

$$\min_{Q,L,\gamma_q} \gamma_q$$

subject to

$$\begin{bmatrix} -Sym(\mathbb{A}^\lambda) & B_1^\lambda & \mathbb{C}^{\lambda T} \\ * & \gamma_q I & -D_{11}^{\lambda\,T} \\ * & * & \gamma_q I \end{bmatrix} \oplus Q \succ 0, \tag{36}$$

where $\mathbb{A}^\lambda := A^\lambda Q + B_2^\lambda L$ and $\mathbb{C}^\lambda := C_1^\lambda Q + D_{12}^\lambda L$. The constraints are imposed at the vertices of Λ_q. The optimal state feedback is given by

$$F := LQ^{-1}, \tag{37}$$

where $L \in \mathbb{R}^{n_u\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$ are the solutions of the optimization problem in Eq. (36).

2) Output feedback: Fixing F in Eq. (37) as C_{K_q} and $D_{K_q} = 0$ in Eq. (23), we design a full order dynamic output feedback

$$u = \left[\begin{array}{c|c} A_{K_q} & B_{K_q} \\ \hline F & 0 \end{array}\right] y, \quad \text{where } A_{K_q} \in \mathbb{R}^{n\times n} \tag{38}$$

that optimizes $\mathcal{H}_\infty$ performance. Due to Theorem 7 in [5], this problem can be solved by convex optimization with LMIs:

$$\min_{X,Y,Z,G,\gamma_q} \gamma_q,$$

subject to

$$\begin{bmatrix} -Sym(P\tilde{A}_{cl}^\lambda) & P\tilde{B}_{cl}^\lambda & \tilde{C_{cl}^\lambda}^T \\ * & \gamma_q I & -\tilde{D_{cl}^\lambda}^T \\ * & * & \gamma_q I \end{bmatrix} \oplus P \succ 0, \tag{39}$$

where $P := X \oplus Y \in \mathbb{R}^{2n\times 2n}$, $X = X^T \in \mathbb{R}^{n\times n}$, $Y = Y^T \in \mathbb{R}^{n\times n}$, $Z := YA_K \in \mathbb{R}^{n\times n}$ and $G := YB_K \in \mathbb{R}^{n\times n_y}$. Moreover, the matrices $P\tilde{A}_{cl}^\lambda$, $P\tilde{B}_{cl}^\lambda$, $\tilde{C_{cl}^\lambda}$ and $\tilde{D_{cl}^\lambda}$ are defined as follows

$$P\tilde{A}_{cl}^\lambda := \begin{bmatrix} X(A^\lambda + B_2^\lambda F) & -XB_2^\lambda F \\ Y(A^\lambda + B_2^\lambda F) - Z - GC_2^\lambda & Z - YB_2^\lambda F \end{bmatrix},$$

$$P\tilde{B}_{cl}^\lambda := \begin{bmatrix} XB_1^\lambda \\ YB_1^\lambda - GD_{21}^\lambda \end{bmatrix}, \quad \tilde{D_{cl}^\lambda} := D_{11}^\lambda,$$

$$\tilde{C_{cl}^\lambda} := \begin{bmatrix} C_1^\lambda + D_{12}^\lambda F & -D_{12}^\lambda F \end{bmatrix}.$$

Again, the constraints in the LMIs in Eq. (39) are imposed at the vertices of Λ_q. Using the optimizers, A_{K_q} and B_{K_q} are obtained as

$$A_{K_q} := Y^{-1}Z, \quad B_{K_q} := Y^{-1}G. \tag{40}$$

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